Program Package *GRAL* for Solution of Boundary Value Problem for PDE in Domains with Complicated Geometry

GREBENNIKOV ALEXANDRE
Facultad de ciencias Físico Matemáticas
Benemérita Universidad Autónoma de Puebla,
Av. San Claudio y Río verde, Ciudad Universitaria, CP 72570, Puebla
MÉXICO
agrebe50@yahoo.com.mx

Abstract: - Program package *GRAL* in MATLAB system is constructed for realization of new fast General Ray (*GR*) Method for solution of boundary value problems for Poisson type PDE with variable coefficients in arbitrary star domains. General Ray (*GR*) method consists in application of the Radon transform directly to the PDE and in reduction PDE to assemblage of Ordinary Differential Equations (ODE). This version of *GR*-method is justified theoretically and presents the solution of the boundary value problem by explicit analytical formulas. We present some formulas, a scheme of *GR*-algorithm, description of *GRAL* program package and results of some numerical experiments that demonstrate the quality and rapidity of constructed *GRAL* package in comparison with *pdemodel* program of PDE toolbox of MATLAB system. *GRAL* package presents grand opportunity for fast solution of many applied problems and it serves also as computerized element in the educative process at courses of lectures, dedicated to PDE, in the postgraduate level at the Faculty of Physical-Mathematical Sciences in Merited Autonomous University of Puebla, Mexico.

Key-Words: - Program package, MATLAB system, partial differential equations.

1 Introduction

There are two main approaches for solving boundary value problems for partial differential equations in analytical form: the Fourier decomposition and the Green function method [1]. The Fourier decomposition is used, as the rule, only in theoretical investigations. The Green function method is the explicit one, but it is difficult to construct the Green function if the considered domain Ω has the complex geometry. The known numerical algorithms are based on the Finite Differences method, Finite Elements (Finite Volume) method and the Boundary Integral Equation method. Numerical approaches lead to solving systems of linear algebraic equations [2] that require a lot of computer time and memory.

We consider here elliptic partial differential equation $div\left[\varepsilon(x,y)\, grad\ u(x,y)\right]=\psi(x,y)$, where $\psi(x,y)$, $\varepsilon(x,y)$ are known functions, in arbitrary star domain Ω. A new approach for the solution of the Dirichlet boundary value problems on the base of the General Ray Principle (*GRP*) was proposed in [3] for the stationary waves field. *GRP* leads to General Ray (*GR*) method, which consists in application of the Radon transform [4] directly to the PDE and in reduction PDE to assemblage of Ordinary Differential Equations (ODE). This version of *GR*-method is justified theoretically in [5], [6], realized as algorithms and program package in MATLAB system, illustrated by numerical experiments. *GR*-algorithm presents the solution of the Dirichlet boundary value problem for the Poisson-type equation with variable coefficients by explicit analytical formulas, using the fast realization of inverse Radon transform due the Fast Fourier Discrete transform.

Here we present a scheme of *GR*-algorithm, some formulas, description of *GRAL* program package and some tests for the constructed programs in comparison with *pdemodel* program of PDE toolbox of MATLAB system.

2 *GR*-Algorithm

Let us consider the Dirichlet boundary problem for the equation:

$$div\left[\varepsilon\left(x,y\right) grad\ u\left(x,y\right)\right] = \psi(x,y), \quad (x,y)\in\Omega; \tag{1}$$

$$u\left(x,y\right) = f(x,y), \quad (x,y)\in\Gamma, \tag{2}$$

with respect to the function $u(x,y)$ that has two continuous derivatives on bought variables inside the plane domain Ω, bounded with a continuous curve Γ. Here $\psi\left(x,y\right)$, $\varepsilon\left(x,y\right)$, $(x,y)\in\Omega$ and $f(x,y)$, $(x,y)\in\Gamma$ are given functions.

The scheme of the *GR*-algorithm can be explained as the consequence of the next steps:

1) reduce the boundary value problem to homogeneous one with respect the function $u_0(x,y)$;
2) reduce the equation (1) with variable coefficient to equivalent equation with constant coefficient with respect the function $v\left(x,y\right)$;
3) describe the distribution of the potential function $v\left(x,y\right)$ along the general ray (a straight line l) by its direct Radon transform $\hat{v}\left(p,\varphi\right)$;
4) construct the family of ODE on the variable p with respect the function $\hat{v}\left(p,\varphi\right)$;
5) resolve the constructed ODE with the zero boundary conditions;
6) calculate the inverse Radon transform of the obtained solution reconstructing $v\left(x,y\right)$;
7) reconstruct $u_0(x,y)$, using functions $v\left(x,y\right)$ and $\varepsilon\left(x,y\right)$;
8) regress to the initial boundary conditions.

We present bellow some formulas, which we use to realize this scheme.

We suppose that the boundary Γ can be described in the polar coordinates (r,α) by some one-valued positive function that we denote $r_0(\alpha)$, $\alpha\in[0,2\pi]$. It is always possible for the simple connected star region Ω with the centre at the coordinate origin. Let us write the boundary function as

$$\bar{f}\left(\alpha\right) = f(r_0(\alpha)\cos\alpha, r_0(\alpha)\sin\alpha). \tag{3}$$

Supposing that functions $r_0(\alpha)$ and $\bar{f}\left(\alpha\right)$ have the second derivative, we introduce functions

$$f_0(\alpha) = \bar{f}(\alpha)/r^2{}_0(\alpha), \quad (x,y)\in\Omega. \tag{4}$$

$$u_0\left(x,y\right) = u\left(x,y\right) - r^2 f_0(\alpha). \tag{5}$$

To realize the first step of the scheme we can write the boundary problem with respect the function $u_0\left(x,y\right)$ as the next two equations:

$$div\left(\varepsilon\nabla u_0\right) = \psi_0\left(x,y\right), \quad (x,y)\in\Omega; \tag{6}$$

$$u_0\left(x,y\right) = 0, \quad (x,y)\in\Gamma. \quad , \tag{7}$$

where function ψ_0 in polar coordinates can be written as

$$\psi_0 = \psi - 4\varepsilon f_0(\alpha) - 2rf_0(\alpha)\left(\frac{\partial\varepsilon}{\partial r}\right) - \varepsilon f_0''(\alpha) - f_0'(\alpha)\left(\frac{\partial\varepsilon}{\partial\alpha}\right).$$

To realize the second step of the scheme, i.e. to reduce the equation (6) with variable coefficient to equivalent equation with constant coefficient, we make substitution:

$$\varepsilon\left(\frac{\partial u_0}{\partial x}\right) = \left(\frac{\partial v}{\partial x}\right) \quad , \quad \varepsilon\left(\frac{\partial u_0}{\partial y}\right) = \left(\frac{\partial v}{\partial y}\right) \tag{8}$$

and obtain corresponding equation:

$$\Delta v = \psi_0, \tag{9}$$

which supplement with boundary condition:

$$v(x, y) = 0, \quad (x, y) \in \Gamma. \tag{10}$$

Using the Kellogg theorem [7] we obtain equivalence of the problem (6)-(7) to the problem (9)-(10), that is the basic element of the theoretical justification of *GR*-method for considering class of equations. The next steps of *GR*-algorithm are applied to the equation (9) with constant coefficient, which are described sufficiently in works [5], [8] and guarantee fast calculation of function $v(x, y)$. In [6] there are presented formulas, based on relations (8), that give possibility for convex domains to reconstruct function $u_0(x, y)$ using calculated function $v(x, y)$.

3 Description of the *GRAL* Program Package

We have constructed the fast program realization of developed algorithms for *GR*-method as the *GRAL* program package in MATLAB system. The package consists of the set of programs, that includes 5 main blocks:

1) programs *R(AL), F(AL)* , *EP(x,y), PSI(x,y)* that realize functions $r_0(\alpha)$, $\bar{f}(\alpha)$, $\varepsilon(x, y), \psi(x, y)$, and must be constructed by user;

2) calculation of the direct and inverse Radon transform with programs *GRAD* and *GIRAD* that present the original modification of the MATLAB programs *radon, iradon* [6];

3) block that corresponds to realization of *GR*-algorithm for constant coefficient ε ;

4) block for variable double smooth coefficient ε ;

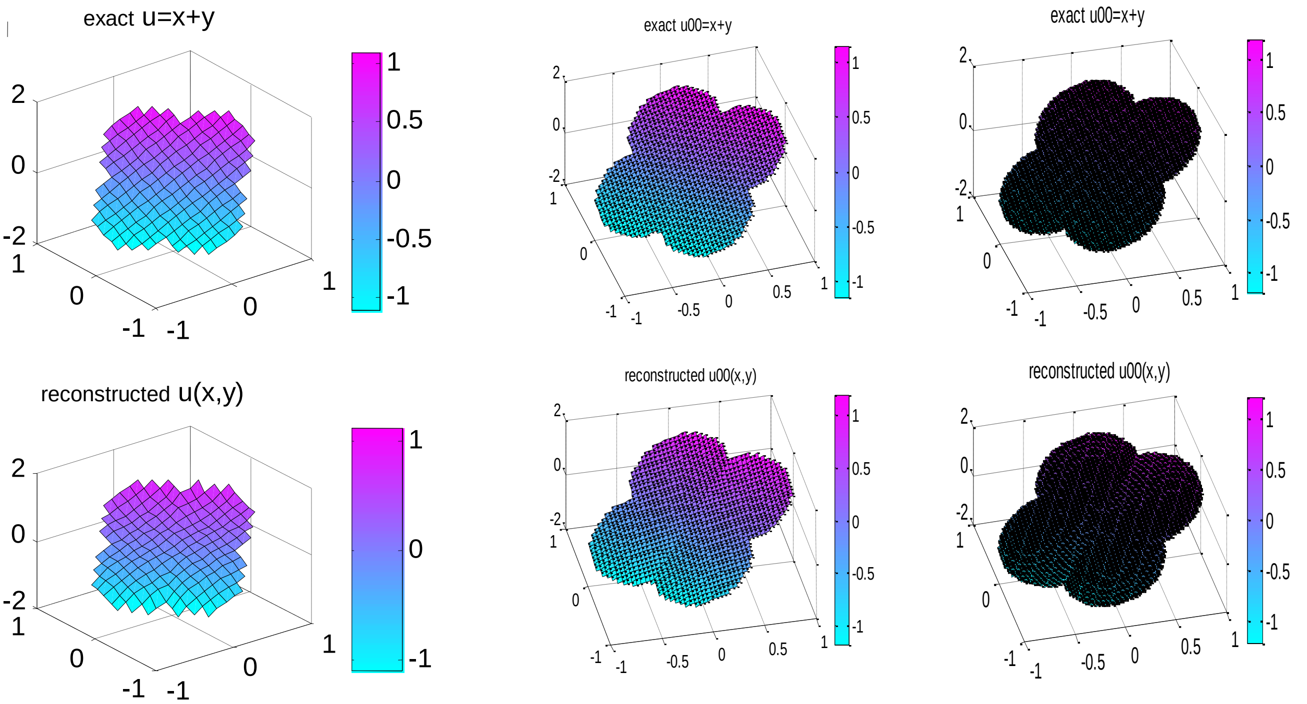
5) block for piecewise constant coefficient ε ;

The main programs in every block 3) -5) realize calculation on the bi-dimensional rectangular red with *NxN* nodes in the minimal rectangle that contains the domain Ω. The function $u(x, y)$ is calculated in such nodes that belong to the domain Ω. The main programs present also graphic illustration of the approximate solution using corresponding MATLAB programs. The block 5) is innovation that yet is not justified theoretically, but we demonstrate its validity by numerical experiments.

The perspective development of the *GRAL* program package will include solution of the boundary problems for the Helmholtz equation, parabolic and hyperbolic equations, so as solution of inverse coefficient problems for PDE with spline regularization [9].

4 Results of Numerical Experiments

We made tests on mathematically simulated model examples with known exact functions $u(x,y)$, $f(x,y)$, $\psi(x,y)$, $\varepsilon(x,y)$. The first example corresponds to the constant coefficient $\varepsilon = 1$, $u(x,y) = x + y$, domain and results of calculation are presented at Figure 1.



N=11 N= 33 N=63

Figure 1.

Some other results for constant coefficient ε are presented in [5], [8]. We present here some new examples for variable ε.

The second example corresponds to $\varepsilon(x,y) = \dfrac{1}{\cos(x+y)+2}$, $u(x,y) = \sin(x+y) + 2(x+y)$ As domain, we have in this case the unit circle. This example is devoted to comparing of *GR*-method (*GRAL* package) with Finite Elements method (*pdemodel* program of PDE toolbox of MATLAB). The results are presented at Table 1 and Figure 2.

In Table 1 we present for corresponding ***N*:** time of calculation by *pdemodel* (the 2-nd column) and by *GRAL* (the 3-d column). In the two next columns we see maximum and mean errors of approximation exact $u(x,y)$ by GRAL.

Table 1

N	***pdemodel***	***GRAL***	**Maximum error**	**Mean error**
14	0.902346sec	0.32455sec	0.6555	0.1713
28	0.869951sec	0.097181sec	0.4917	0.1076
42	2.387995sec	0.197593sec	0.3985	0.0819

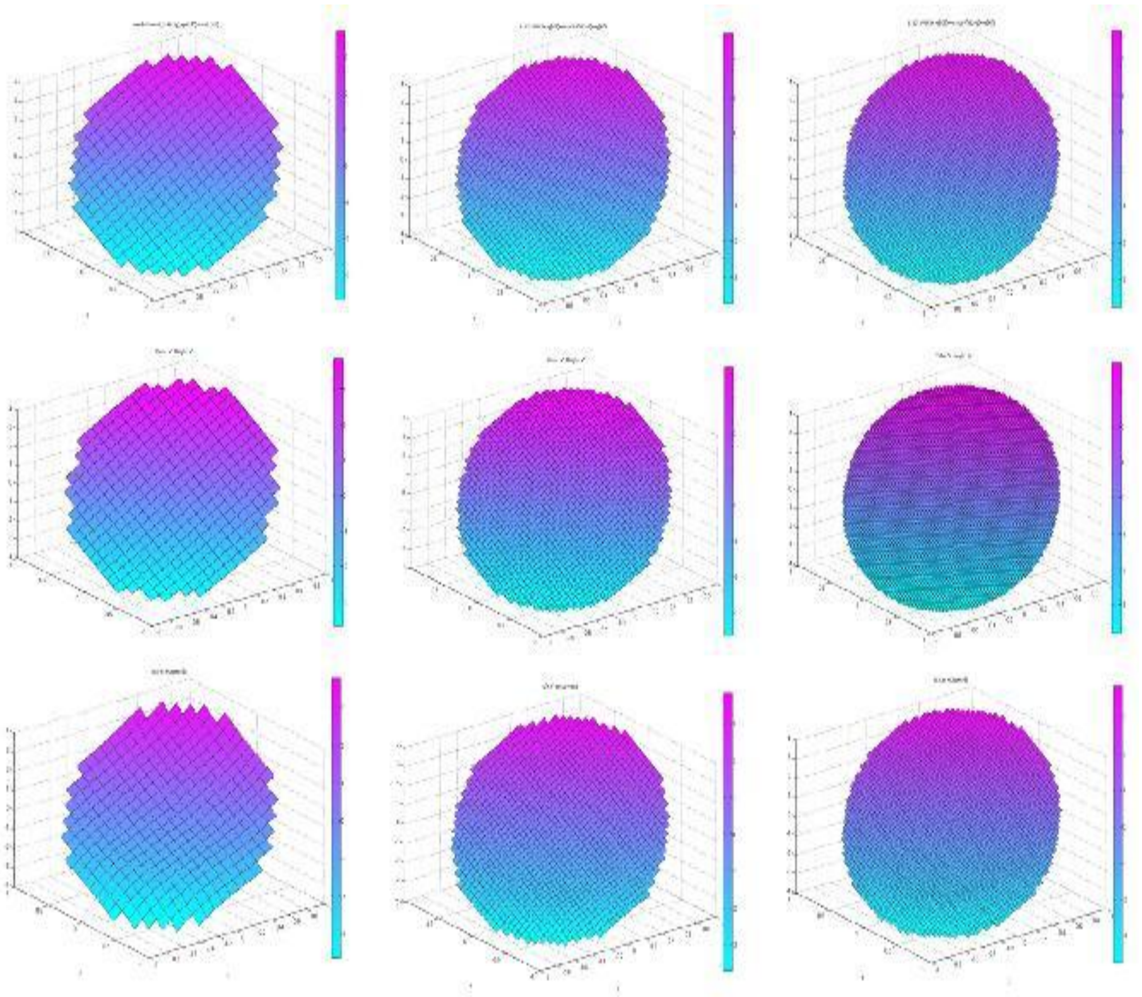


Figure 2.

In Figure 2 there are shown: in the 1-st, 2-nd and 3-d columns contain results for N=14, 28 and 42 correspondently; in the 1-st line – exact $u(x, y)$, the 2-nd line – $u(x, y)$ reconstructed by *pdemodel* , the 3-d line – $u(x, y)$ reconstructed by *GRAL*.

Analysis of Figure 2 and Table 1 demonstrates that *GR*-method (*GRAL* package) guarantees a good approximation and require sufficiently less time of calculations in comparing with Finite Elements method (*pdemodel* program of PDE toolbox of MATLAB).

In the third example we have $\psi(x, y) = 0,$ piecewise constant coefficient ε :

$$\varepsilon(x,y) = \begin{cases} 2, & y \geq -x; \\ 1, & y < -x; \end{cases}$$

and the next exact solution

$$u(x,y) = \begin{cases} x + y, & y \geq -x; \\ 2(x+y), & y < -x. \end{cases}$$

At Figure 3 , graph (a), we can see exact $u(x, y)$ and at graph (b) – solution, reconstructed by *GRAL*.

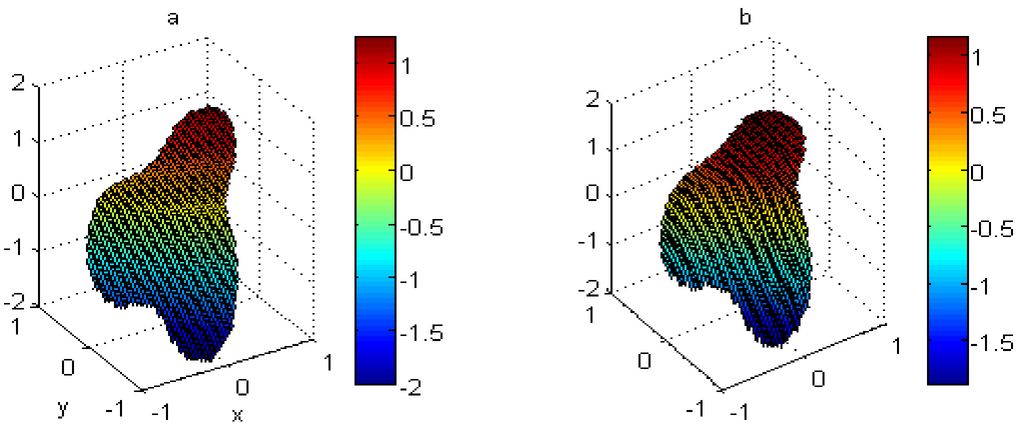


Figure 3.

5 Acknowledges

The author acknowledges to CONACYT Mexico, Merited Autonomous University of Puebla (MAUP) Mexico and Aerospace School of Moscow Aviation Institute for approval of this investigation in the frame of the Probation Period at 2012 – 2013 years. The author bless also to VIEP of MAUP, Mexico for the support of the part of this investigation in the frame of the Project No GRA-EXC12-I.

References:

[1] S. L. Sobolev, *Equations of Mathematical Physics*, Moscow, 1966.

[2] A.A. Samarsky, *Theory of Difference Schemes*, Moscow, 1977.

[3] Alexander Grebennikov, General Ray Method for Solution of Boundary Value Problems for Elliptic Partial Differential Equations. APLIEDMATH III. Congreso Internacional en Matemáticas Aplicadas, Instituto Politécnico Nacional, México, 2007, pp. 200-209.

[4] Helgason Sigurdur, *The Radon Transform*, Birkhauser, Boston-Basel-Berlin, 1999.

[5] Alexander Grebennikov, General Ray Method for Solution of Dirichlet Boundary Value Problems for Poisson Equation in Arbitrary Simple Connected Star Domain. Proceedings of 2-nd International Symposium “Inverse Problems, Design and Optimization”. Miami, Florida, U.S.A., pp. 101-106, 2007.

[6] Emmanuel Abdias Romano Castillo, Alexandre I. Grebennikov, Justificación analítica y numérica del método de solución del problema directo $div\left(\varepsilon\left(x,y\right)\nabla u\right)=0$, con coeficiente variable y condición de contorno tipo Dirichlet, *Boletín de la Sociedad Cubana de Matemática y Computación*, Vol. 9, No. 1, Abril, 2001, pp. 33-46.

[7] David Gilbarg, Neil S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 1998.

[8] A. Grebennikov, R. Paredes Jaramillo, Program Realization of *GR*-Algorithm and It’s Numerical Experimental Comparison on Rapidity with *Pdemodel* Program of Matlab System. Proceedings of IV International Conference "Computer Modeling 2008", Sanct-Petersburg, Russia, 26-27, June 2008, pp. 67-71.

[9] A. Grebennikov, *Spline Approximation Method and Its Applications*, MAX Press., Rus., 2004.