

PATH GENERATION METHODS FOR VALUATION OF LARGE VARIABLE ANNUITIES PORTFOLIO USING QUASI-MONTE CARLO SIMULATION

Ben Mingbin Feng

Kai Liu

Department of Statistics and Actuarial Science
University of Waterloo
200 University Ave W
Waterloo, ON, CANADA

School of Mathematical and Computational Sciences
University of Prince Edward Island
550 University Avenue
Charlottetown, PE, CANADA

ABSTRACT

Variable annuities are long-term insurance products that offer a large variety of investment-linked benefits, which have gained much popularity in the last decade. Accurate valuation of large variable annuity portfolios is an essential task for insurers. However, these products often have complicated payoffs that depend on both of the policyholder's mortality risk and the financial market risk. Consequently, their values are usually estimated by computationally intensive Monte Carlo simulation. Simulating large numbers of sample paths from complex dynamic asset models is often a computational bottleneck. In this study, we propose and analyze three Quasi-Monte Carlo path generation methods, Cholesky decomposition, Brownian Bridge, and Principal Component Analysis, for the valuation of large VA portfolios. Our numerical results indicate that all three PGMs produce more accurate estimates than the standard Monte Carlo simulation at both the contract and portfolio levels.

1 INTRODUCTION

Variable annuities (VAs) are investment-linked long term insurance products that represent an important source of retirement income in many countries. In the United States, the sales estimates for variable annuity contracts exceed \$100 billion in 9 of 10 years in the past decade, peaking at \$158 billion in 2011 (Secure Retirement Institute 2020). In contrast to traditional fixed annuities, the premium of a variable annuity is invested in a segregated fund and the contract benefits are linked to the fund performance; these benefits can often be viewed as embedded financial options. Hardy (2003) argues that VAs can be modeled as insurance products with exotic embedded options. These exotic options offer customized benefits for different policyholders' needs, which partially contribute to the popularity of VAs. Often the premiums of all VAs in a portfolio are invested in segregated funds with the same underlying assets, so diversification is minimal. Therefore, it is crucially important for insurers to monitor and manage the significant financial risks involved in large VA portfolios. In other words, estimating the distribution of the future values of VA portfolios and quantifying the associated financial risks is a pressing need for insurers.

Estimating the value of a VA portfolio often requires valuation of individual VA contracts. For a VA contract with exotic embedded options, Monte Carlo (MC) can sometimes be the only viable way to estimate its value. Yet running MC for hundreds of thousands of VA contracts can be a prohibitive computational burden. There have been significant research efforts in recent years devoted to addressing the heavy computational burden of MC for evaluation and risk management of large VA portfolios. One main research venue, inspired by Gan (2013), is to save computations by running simulations for only a selective set of *representative contracts*. Then their estimated values are used to calibrate a predictive model, which is then used to predict the other contracts' values. This three-components simulation framework is later summarized by Feng et al. (2020): (1) A compressor that selects representative contracts, (2) a

simulator that runs simulations for the representative contracts, and (3) a predictor that estimates the values of other contracts and the value of the entire portfolio. Gan (2013) and Gan and Lin (2015) both clustering, standard MC, and Gaussian process model as the three components, with nugget effects added in the latter study. Gan and Lin (2017) also use clustering as the compressor, but a dynamic hedging simulator and a two-level Gaussian process predictor are considered. Hejazi and Jackson (2016), Hejazi et al. (2017), and Gan and Valdez (2018) use neural network, spatial interpolation, and regression models, respectively, as the predictors. Xu et al. (2018) use a moment matching scheme as the compressor and a few machine learning models such as neural networks, regression trees, and random forest as the predictors. Liu and Tan (2020) apply Quasi-Monte Carlo method in the compressor and a Taylor approximation scheme in the predictor. Feng et al. (2020) added a optimum budget allocation in the simulator to strategically allocate a fixed simulation budget to different contracts. In the numerical studies in Liu and Tan (2020) and Feng et al. (2020), some of these methods poorly estimates the VA contract values, but the estimation errors at the contract level cancel out to some degree when aggregated to the portfolio level. Low contract-level accuracy is a major drawback for using the three-components simulation framework, because the simulation is only run for a small number of VA contracts in the portfolio.

In this study, we consider a simulation method that runs simulation for all VA contracts in a portfolio, but save computations by simulating a small number of sample paths. Such method produces accurate estimates at both the portfolio and the contract levels. Specifically, we consider the path generation methods (PGMs) in Quasi-Monte Carlo (QMC) so that each contract value can be accurately estimated with a small number of sample paths. As such, we can afford running simulations for all VAs in a large portfolio, resulting in highly accurate contract and portfolio value estimates. QMC has been a strong contender to standard MC since its introduction, for both its high convergence rate in some cases and its good practical performances in many applications. In essence, QMC concerns about the design and analysis of deterministic sequences in the high dimensional unit cube and their applications in numerical integration. Well-designed QMC sequences that satisfy certain uniformity properties can densely fill the high dimensional unit cube, allowing the numerical integration quickly converges to its true value. Interested readers are encouraged to refer to Niederreiter (1992) for a thorough introduction to QMC. Unlike standard MC that converges inversely proportionally to the number of independent replications, the convergence rate of the QMC depends on not only the number of QMC points but also the dimensionality of the problem. Sloan and Woźniakowski (1998) and Sloan and Woźniakowski (2001) show that there exist QMC algorithms where the curse of dimensionality can be alleviated in some weighted function classes. Papageorgiou (2000) and Owen (2003) both demonstrate the superiority of QMC in some isotropic integrals. In contrast to theoretical predictions, Bratley et al. (1992) show that without considering the function of interest, QMC may offer no practical advantage over MC.

Path generation methods (PGMs) concern the transformation of QMC sequence to other random variables. It has been studied that PGMs have significant impacts on the efficiency of QMC algorithms. For example, Wang and Tan (2012) study different PGMs in financial applications and found that the accuracies of the resulting QMC algorithms are vastly different, even for the same problem. Wang and Tan (2013) extend these findings to discontinuous pricing functions and demonstrated highly accurate option price estimates with minimal computations. In addition, Liu (2012) summarize a general derivative pricing framework using PGMs. Inspired by the above studies, in this article we investigate three common PGMs for the valuation of large VA portfolios.

The remaining paper is organized as follows. Section 2 provides a brief overview of variable annuity valuation. Section 3 introduces the path generation methods. Section 4 discusses the numerical evidence and Section 5 concludes the paper.

2 VARIABLE ANNUITY VALUATION

Under appropriate mathematical and financial assumptions, such as the existence and uniqueness of risk-neutral measure and market completeness, MC can be used to estimate the value of various financial instruments such as exotic options. Since the seminal work Boyle (1977), MC has become a popular method

for option pricing (Boyle et al. 1997) and many other financial engineering applications (Glasserman 2004). Also, soon after its introduction in Niederreiter (1992), QMC has also become a popular method for option pricing (Joy et al. 1996); see L'Ecuyer (2004) for a survey study on QMC methods in finance. In this section, we will articulate the problem statement of variable annuity valuation. In addition, we will show how different path generation methods can be applied to solve this problem.

Consider a VA contract whose benefit is linked to an underlying fund whose value at time t is denoted by S_t , for any $0 \leq t \leq T$, where T is the maturity of the contract. The fund value is used to calculate the growth of the policyholder's sub-account value, which is denoted by F_t at time t . For simplicity of exposition, we assume that there are no fees so $F_0 = S_0$; the sub-account value can deviate from fund value in the future due to withdrawals. A vanilla guaranteed minimum maturity benefit (GMMB) allows the policyholder to receive the sub-account value at maturity or a guaranteed amount G , whichever is larger. This means that, at maturity T , besides the sub-account value F_T , the insurer is liable to pay $(G - F_T)^+ = \max\{0, G - F_T\}$ to the policyholder. This liability is exactly the payoff of a vanilla European put option. Intuitively, when the underlying fund is not performing well, and sub-account value is insufficient to pay the guaranteed benefit G , then the insurer has to pay the difference. A guaranteed minimum accumulation benefit (GMAB) is similar to GMMBs but contains a few renewal or ratchet points during the contract life when the guaranteed benefit can be renewed to lock in the accumulated benefit up to that point. The insurer liability for a GMAB with one renewal point is similar to the payoff of a put-on-put option; the payoff quickly becomes very complicated as the number of renewal points increases. Guaranteed minimum withdrawal benefit (GMWB) is a type of VA contract that allows the policyholder to withdraw up to a fixed guaranteed amount periodically. When the underlying fund follows a Markovian dynamic asset model, and the policyholder withdraws the maximum amount in each period, Liu (2010) shows that the insurer's liability can be modeled as arithmetic Asian options. Typically a VA contract also provides guaranteed minimum death benefit (GMDB) to pay the beneficiary, upon the policyholder's death, the maximum of the sub-account value and the guaranteed amount. These benefits, as well as their variations, can be combined in a single VA contract, resulting in even more complicated payoff structures and exotic embedded options. Monte Carlo simulation is often the only viable method to estimate the value of a VA contract.

To make the case concrete, we present the simulation model for the actuarial present value of a VA contract with both GMDB and GMWB. This simulation model is inspired by (Gan 2013) and is employed in our numerical studies in Section 4. Let $t = 0$ be the inception of a VA contract and T be the maturity of the contract. Also, let $t = 0, 1, 2, \dots, T$ be the anniversary dates of the contract, when one of two possible events may happen: (1) the policyholder withdraws money as a guaranteed withdrawal of the GMWB or (2) the policyholder dies. We use subscripts t^- and t^+ to denote the value of a variable (e.g., F_{t^-} and F_{t^+}) immediately before and after an anniversary date, respectively. We assume for simplicity that there are no fees, no lapses (abrupt termination of the contract), the policyholder takes maximum annual withdrawals, and all the events happen only at anniversary dates. Noted that these assumptions can be relaxed with some revisions to the simulation model. Also, the simulation model can be modified to accommodate other event frequencies such as monthly and weekly or irregular event times, but we use terms such as "anniversary" and "years" for ease of exposition.

Given the initial fund value S_0 , the first step in the simulation model is to simulate its future values $\mathbf{S} = \{S_1, \dots, S_T\}$ based on some prescribed dynamic asset models, e.g., the Black-Scholes model. At time $t = 0$, the policyholder pays premium F_0 , which is invested into the sub-account. The account value's growth between two anniversary dates is linked to the growth of the underlying fund, i.e.,

$$F_{t^-} = F_{(t-1)^-} \frac{S_t}{S_{t-1}}, \quad t = 1, \dots, T. \tag{1}$$

The GMWB allows the policyholder to withdraw a fraction, say α , of the initial premium every year, with a guaranteed withdrawal amount equal to the initial premium. Let W_t be the withdrawal benefit, D_t be the death benefit, G_t^W be the remaining total amount that can be withdrawn, G_t^E be the maximum

amount that can be withdrawn annually and G_t^D be the guaranteed minimum death benefit at time t . In our simplified settings, the maximum amount that can be withdrawn annually is fixed $G_t^E = G^E = \alpha F_0$. The initial guaranteed total withdrawal is $G_0^W = F_0$ and this amount decreases every time a withdrawal is made; the remaining total amount that can be withdrawn after the t -th withdrawal is denoted by G_{t+}^W . Assuming that the policyholder takes the maximum withdrawal at each anniversary date, the withdrawal amount at year t is then $E_t = \min(G^E, G_{t-}^W)$ for $t = 1, \dots, T$. If the sub-account value at year t is insufficient for the withdrawal, the insurer is liable for the difference, which is

$$W_t = \max\{0, E - F_{t-}\}, \quad t = 1, \dots, T.$$

After this t -th withdrawal, the sub-account value becomes $F_{t+} = \max(0, A_{t-} - E_t)$. Also the remaining total amount that can be withdrawn is $G_{t+}^W = \max(0, G_{t-}^W - E)$; this amount does not change between two consecutive withdrawals, i.e., $G_{(t+1)-}^W = G_{t+}^W$.

The GMDB guarantees a minimum death benefit, denoted by G_t^D , that is initially set equal to the initial premium and subsequently adjusted pro rata to the sub-account value as withdrawals are made. Mathematically, this means that $G_0^D = F_0$ and $G_{t+}^D = G_{t-}^D \frac{F_{t+}}{F_{t-}}$, $t = 1, \dots, T$. Also, this death benefit remains unchanged between two consecutive withdrawals, i.e., $G_{(t+1)-}^D = G_{t+}^D$. If the policyholder dies at year t and the sub-account value is insufficient, the insurer is liable to the difference, which is

$$D_t = \max\{0, G_{t-}^D - F_{t-}\}, \quad t = 1, \dots, T.$$

Suppose the policyholder is at age x_0 at time 0, the actuarial present value of the insurer's liability for the death and the withdrawal benefits for given future fund values $\mathbf{S} = (S_1, \dots, S_T)$ can be written as

$$V(\mathbf{S}) = \sum_{t=1}^T ({}_{t-1}p_{x_0}) \cdot (1 - q_{x_0+t-1}) \cdot W_t e^{-rt} + \sum_{t=1}^T {}_{t-1}(p_{x_0}) \cdot (q_{x_0+t-1}) \cdot D_t e^{-rt}, \quad (2)$$

where ${}_t p_x$ is the standard actuarial notation for the probability of aged x surviving for t years, q_x denotes the probability of aged x dying within 1 year, and r is the per-period risk-free interest rate. In our numerical studies, these probabilities are based on the 1996 IAM mortality tables provided by the Society of Actuaries (Johansen 1996). As we alluded to in the notation $V(\mathbf{S})$, the present value (2) depends on the simulated future fund values. An insurer is often interested in the expected value $E[V(\mathbf{S})]$, i.e., the value of the VA contract. This expectation is complicated and has no analytical formula, but one can easily translate the descriptions between (1) and (2) into a simulation computer program and estimate $E[V(\mathbf{S})]$ via simulation. In a MC experiment, the user simulates independent sample paths $\mathbf{S}^{(1)}, \dots, \mathbf{S}^{(N)}$ and estimates the VA contract value by the sample average of the resulting present values $V(\mathbf{S}^{(1)}), \dots, V(\mathbf{S}^{(N)})$.

3 PATH GENERATION METHODS (PGMs)

The expected VA contract value described in Section 2, i.e., $E[V(\mathbf{S})]$, has no closed-form analytical formula, thus it needs to be estimated numerically by, say, MC or QMC. The simulation efficiency depends on how well one can simulate the random sample path \mathbf{S} according to the prescribed dynamic asset model. Three PGMs in QMC methods are presented in this section. For clarity of illustration, we focus our discussions on a single-asset Black-Scholes model for the underlying fund value, but the PGMs in our discussions are generally applicable to other asset models and to multiple-asset models.

Suppose the underlying fund value S_t follows the Black-Scholes model, then its risk-neutral instantaneous asset dynamic at any time $0 < t < T$ is given by

$$dS_t = rS_t dt + \sigma S_t dW_t, \quad (3)$$

where r is the per-period risk-free interest rate, σ is the per-period volatility, and W_t is a standard Brownian motion. Given an initial value S_0 , one solution to the stochastic differential equation (3) is $S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$. So the growth of the fund value between two times depends on the corresponding increment of W_t , i.e., $S_{t_j} = S_{t_i} \exp\left(\left(r - \frac{\sigma^2}{2}\right)(t_j - t_i) + \sigma(W_{t_j} - W_{t_i})\right)$ for any $0 \leq t_i < t_j \leq T$. Since W_t is a standard Brownian motion, it has independent normally distributed increments, i.e., $W_t - W_s \sim \mathcal{N}(0, t - s)$ for any $0 \leq s < t \leq T$. The covariance of the Brownian motion at any two given times $t, s \in [0, T]$ is given by $Cov[W_t, W_s] = \min(t, s)$. So, for fixed times $0 = t_0 < t_1 < \dots < t_K = T$, the sample path of the fund value can be simulated as

$$S_{t_i} = \exp(\mu_i + \sigma B_{t_i}), \text{ where } \mu_i = \ln S_0 + (r - \sigma^2/2)t_i. \tag{4}$$

The random vector $\mathbf{B} = [B_{t_1}, \dots, B_{t_K}]$ follows a multivariate normal distribution with zero means and covariance matrix $\mathbf{\Sigma}$ whose (i, j) -th element is $\Sigma_{ij} = \min(t_i, t_j)$; we write $\mathbf{B} \sim MVN(\mathbf{0}, \mathbf{\Sigma})$. In light of the one-to-one relationship (4) between \mathbf{S} and \mathbf{B} , VA contract value $V(\mathbf{S})$ in (2) can be redefined as $\tilde{V}(\mathbf{B})$ for some function $\tilde{V}(\cdot)$. In subsequent discussions, we will turn our attention to the efficient generation of \mathbf{B} .

In a typical high dimensional QMC method, a point sequence in the unit hypercube $[0, 1]^K$ is generated. For each QMC point, the value in each dimension can be plugged into the inverse cumulative distribution function (cdf) of a standard normal random variable. The resulting random vector, denoted by \mathbf{Z} , can be viewed as a QMC sequence of standard multivariate normal random vector.

Let $\mathbf{Z} \sim MVN(\mathbf{0}, \mathbf{I}_K)$, where \mathbf{I}_K is the $K \times K$ identity matrix, be a standard multivariate normal random vector. Consider a linearly transformed random vector $\mathbf{B} = \mathbf{AZ}$ for a given $K \times K$ matrix \mathbf{A} , it is well-known in statistics that $\mathbf{B} \sim MVN(\mathbf{0}, \mathbf{AA}^\top)$. If the matrix \mathbf{A} satisfies $\mathbf{AA}^\top = \mathbf{\Sigma}$, then $\mathbf{B} \sim MVN(\mathbf{0}, \mathbf{\Sigma})$ and \mathbf{A} is known as the generating matrix of the Brownian motion W_t . Different PGMs correspond to different ways to construct the generating matrix. Note that the efficiency of standard MC is not affected by the generating matrix, as long as $\mathbf{AA}^\top = \mathbf{\Sigma}$. However, well-designed PGMs can greatly improve the efficiency of QMC methods, as shown in Wang and Tan (2012). We study three PGMs in this study: Cholesky decomposition (Section 3.1), Brownian bridge (Section 3.2), and principal component analysis (Section 3.3). In addition, when the decomposition matrix \mathbf{A} defines the affine transformation of any marginal distributions. The PGMs could be applied to more complicated models, e.g. general hyperbolic models.

3.1 Cholesky Decomposition

Cholesky decomposition is the most common PGM for simulating Brownian motions, in both MC and QMC. Specifically, the generating matrix for the Cholesky PGM, denoted by \mathbf{A}^{CHO} , is simply the Cholesky decomposition of the covariance matrix $\mathbf{\Sigma}$, which is given by

$$\mathbf{A}^{CHO} = \begin{bmatrix} \sqrt{t_1} & 0 & \dots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \dots & \sqrt{t_K - t_{K-1}} \end{bmatrix}.$$

Note that the Cholesky PGM $\mathbf{B} = \mathbf{AZ}$ is exactly the random walk construction of Brownian motions:

$$B_{t_i} = B_{t_{i-1}} + \sqrt{t_i - t_{i-1}}Z_i, \text{ for } i = 1, \dots, K. \tag{5}$$

The popularity of the Cholesky PGM mainly owes to its simplicity as the random walk construction closely resembles the stochastic differential equation for X_t . Also, computationally, since the Cholesky generating matrix \mathbf{A}^{CHO} is lower-triangular, $\mathbf{B} = \mathbf{A}^{CHO}\mathbf{Z}$ requires only $\mathcal{O}(K)$ operations, instead of $\mathcal{O}(K^2)$ operations for matrix multiplication \mathbf{AZ} in general.

3.2 Brownian Bridge

The random walk construction (5) simulates B_{t_i} , $i = 1, \dots, K$ in order. However, one may simulate the values B_{t_i} , $i = 1, \dots, K$ in any order, provided that the random vector \mathbf{B} follows the desired joint distribution, i.e., $\mathbf{B} \sim MNV(\mathbf{0}, \Sigma)$. For example, one may first simulate the end value $B_T = B_{t_K}$ given the initial value $B_0 = B_{t_0}$, and then simulate the mid-point $B_{T/2}$ from the correct conditional distribution of $B_{T/2}$ given B_0 and B_T . This is the main idea of the so-called Brownian bridge (BB) construction.

The BB construction was first proposed by Moskowitz and Caflisch (1996), which simulates Brownian motion sample paths by first simulating the final value of a path then repeatedly simulate the intermediate values conditional on the values that have been simulated. Mathematically, given B_0 , BB first simulates $B_{t_K} = \sqrt{t_K}Z_1 = \sqrt{T}Z_1$. Next, given two adjacent values B_{t_i} and B_{t_k} , the intermediate value B_{t_j} for any $t_j \in (t_i, t_k)$ is simulated as

$$B_{t_j} = (1 - \alpha)B_{t_i} + \alpha B_{t_k} + \sqrt{\alpha(1 - \alpha)(t_k - t_i)}Z, \text{ where } \alpha = \frac{t_j - t_i}{t_k - t_i} \text{ and } Z \sim \mathcal{N}(0, 1).$$

For instance, suppose K is a power of 2 and t_0, t_1, \dots, t_K are equally spaced in $[0, T]$, then Brownian motion sample path simulated by Brownian bridge PGM is

$$\begin{aligned} B_{t_K} &= \sqrt{T}Z_1, \\ B_{t_{K/2}} &= \frac{1}{2}(B_{t_0} + B_{t_K}) + \sqrt{\frac{T}{4}}Z_2 = \frac{\sqrt{T}}{2}Z_1 + \frac{\sqrt{T}}{2}Z_2, \\ B_{t_{K/4}} &= \frac{1}{2}\left(B_{t_0} + B_{t_{K/2}}\right) + \sqrt{\frac{T}{8}}Z_3 = \frac{\sqrt{T}}{4}Z_1 + \frac{\sqrt{T}}{4}Z_2 + \sqrt{\frac{T}{8}}Z_3, \\ B_{t_{3K/4}} &= \frac{1}{2}\left(B_{t_{K/2}} + B_{t_K}\right) + \sqrt{\frac{T}{8}}Z_4 = \frac{3\sqrt{T}}{4}Z_1 + \frac{\sqrt{T}}{4}Z_2 + \sqrt{\frac{T}{8}}Z_4, \\ &\vdots \end{aligned} \tag{6}$$

where Z_i 's are independent standard normal random variables for $i = 1, \dots, K$. Brownian bridge construction is often done recursively as shown in (6), which also corresponds to a specific generating matrix \mathbf{A}^{BB} such that $\mathbf{A}^{BB}(\mathbf{A}^{BB})^\top = \Sigma$. For example, if $K = T = 4$, then according to (6) the Brownian bridge generating matrix is

$$\mathbf{A}^{BB} = \begin{bmatrix} 1/2 & 1/2 & \sqrt{2}/2 & 0 \\ 1 & 1 & 0 & 0 \\ 3/2 & 1/2 & 0 & \sqrt{2}/2 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

Computationally, \mathbf{A}^{BB} is not triangular in general, so the Brownian bridge construction can take longer than the random walk construction, especially for sample paths with many intermediate steps. In terms of accuracy, both $\mathbf{A}^{CHO}\mathbf{Z}$ and $\mathbf{A}^{BB}\mathbf{Z}$ have the same joint distribution, although the random vector \mathbf{Z} is weighted differently. In standard MC, elements of \mathbf{Z} are independent standard normal random numbers without any trivial statistical difference, so the different weightings by the generating matrices do not result in significant difference in estimation accuracy. However, as shown in Caflisch et al. (1997), when QMC methods are applied the Brownian bridge construction can produce more accurate results than the random walk construction for financial securities with special payoff structures. This is because the Brownian bridge starts with the end point of the Brownian path and then consequently refines the midpoint of intervals which will reduce so-called effective dimension (Caflisch et al. 1997). So the generating matrix \mathbf{A} plays an important role in QMC methods. In the next section, we present a generating matrix that has provable optimality properties.

3.3 Principal Component Analysis

For many well-known high dimensional QMC sequences, such as Sobol and Halton sequences, the first few components usually have better uniformity properties than other components. Therefore, to develop

an efficient QMC method for simulating $\mathbf{B} = \mathbf{AZ}$, it is desirable to construct a generating matrix \mathbf{A} such that the variance of \mathbf{B} is mostly explained by the first component of \mathbf{Z} . To be specific, note that the random vector $\mathbf{B} = \mathbf{AZ} = Z_1\mathbf{A}_{.1} + \dots + Z_K\mathbf{A}_{.K}$ where $\mathbf{A}_{.k}$ denotes the k -th column of \mathbf{A} . Since all components of \mathbf{Z} have unit variance, the total variance of \mathbf{B} can be shown as $\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.K}\|^2$. Moreover, the total variance contributed by the first k components of \mathbf{Z} can be shown as $\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.k}\|^2$. To concentrate the total variance to as few components of \mathbf{Z} as possible, one should maximize the ratios

$$R_k = \frac{\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.k}\|^2}{\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.K}\|^2}, \quad \text{for all } k = 1, \dots, K. \quad (7)$$

One way to achieve the above objective is via the eigen-decomposition or, equivalently, the principal component analysis (PCA) of the covariance matrix $\mathbf{\Sigma}$. Since $\mathbf{\Sigma}$ is positive definite, it has an eigen-decomposition, say $\mathbf{\Sigma} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$, where $\mathbf{\Lambda}$ is diagonal matrix containing the eigenvalues of $\mathbf{\Sigma}$ and \mathbf{V} is an orthogonal matrix containing the corresponding eigenvectors as its columns. Without loss of generality, we assume that the eigenvalues are sorted in descending order, i.e., $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K$. Define the PCA generating matrix as

$$\mathbf{A}^{PCA} = \mathbf{V}\sqrt{\mathbf{\Lambda}},$$

where $\sqrt{\mathbf{\Lambda}}$ is a diagonal matrix that contains $\sqrt{\lambda_k}$, $k = 1, \dots, K$ in its diagonal. Then the eigen-decomposition identity dictates that $\mathbf{A}^{PCA}(\mathbf{A}^{PCA})^\top = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top = \mathbf{\Sigma}$ so \mathbf{A}^{PCA} is indeed a generating matrix for the desired Brownian motion.

We provide a holistic explanation for how \mathbf{A}^{PCA} achieves the aforementioned concentration of total variance. By construction, $\mathbf{A}^{PCA} = [\sqrt{\lambda_1}\mathbf{V}_{.1}, \dots, \sqrt{\lambda_K}\mathbf{V}_{.K}]$ where $\mathbf{V}_{.k}$ is the k -th eigenvector. Since \mathbf{V} is an orthogonal matrix, $\mathbf{V}_{.k}^\top\mathbf{V}_{.k} = 1$ and therefore $\|\mathbf{A}_{.k}^{PCA}\|^2 = \lambda_k\mathbf{V}_{.k}^\top\mathbf{V}_{.k} = \lambda_k$ for all $k = 1, \dots, K$. Let \mathbf{A} be any generating matrix for $\mathbf{\Sigma}$, then according Equation (7), for every $k = 1, \dots, K$ we have

$$\frac{\|\mathbf{A}_{.1}^{PCA}\|^2 + \dots + \|\mathbf{A}_{.k}^{PCA}\|^2}{\|\mathbf{A}_{.1}^{PCA}\|^2 + \dots + \|\mathbf{A}_{.K}^{PCA}\|^2} = \frac{\lambda_1 + \dots + \lambda_k}{\lambda_1 + \dots + \lambda_K} \geq \frac{\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.k}\|^2}{\|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.K}\|^2},$$

where the inequality holds because $\|\mathbf{A}_{.1}^{PCA}\|^2 + \dots + \|\mathbf{A}_{.K}^{PCA}\|^2 = \|\mathbf{A}_{.1}\|^2 + \dots + \|\mathbf{A}_{.K}\|^2 = \lambda_1 + \lambda_2 + \dots + \lambda_K$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_K$. This justifies the desire maximum concentration of variance.

For the Brownian motion $\{B_t, t \geq 0\}$ in (4), Akesson (1998) showed that the generating matrix $\mathbf{A}^{PCA} = \mathbf{V}\sqrt{\mathbf{\Lambda}}$ where

$$V_{ij} = \frac{2}{\sqrt{2n+1}} \sin\left(\frac{2i-1}{2n+1}j\pi\right) \text{ and } \lambda_i = \left(4n \sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\right)^{-1}, \quad \text{for all } i, j = 1, \dots, K.$$

Computationally, Scheicher (2007) showed that \mathbf{A}^{PCA} can be computed efficiently using the fast sine transform with $\mathcal{O}(K \log(K))$ basic operations.

4 NUMERICAL STUDIES

In this section, we apply QMC methods with different PGMs to estimate the values of large VA portfolios. The results of our study provide some numerical evidences on the superior efficiency of the QMC methods compared to the standard Monte Carlo simulation. In addition, the results also show that the accuracy of the VA portfolio valuation differs for different PGMs. Besides the contract and portfolio value, we also use the infinitesimal perturbation analysis (IPA) method (Glasserman 2004) to estimate the sensitivity of these values to the changes to initial sub-account fund value, i.e., the Deltas (Δ). The Deltas estimation is used to provide additional supporting evidences for the effectiveness of the proposed QMC methods. The settings of our numerical experiences are inspired by Gan (2013).

In our experiments, we simulate synthetic VA portfolios with 100,000 VA contracts. Each contract is specified by 6 attributes, as shown in the left column of Table 1. For each contract in the synthetic portfolio, the value of each attribute is uniformly and randomly sampled from the values and ranges specified in the right column of Table 1. Note that some contracts have only one guaranteed minimum benefit, i.e., GMDB. These contracts can be viewed as contracts with both GMDB and GMWB but has a withdrawal limit of 0 annually. Also, the gender and age of the policyholder affect the survival and death probabilities ${}_t p_x$ and q_x in the contract value calculation, i.e., Equation (2).

Table 1: Variable annuity contract attributes description

Attributes	Values and Ranges
Guarantee type	GMDB only, GMDB & GMWB
Gender	Male, Female
Age (x_0)	20, 21, ..., 60
Initial Premium (F_0)	[10000, 500000]
GMWB withdrawal rate (α)	0.04, 0.05, ..., 0.08
Maturity (T)	10, 11, ..., 25

We assume that the sub-accounts of all the VA contracts are linked to the same underlying fund, which is modeled by the Black-Scholes model, i.e., geometric Brownian motion. In practice, the sub-accounts are all linked to funds that consist of a small number of indices. We consider a single-fund example for illustration purpose. Let $T = 25$ be the maximum maturity of all contracts in the portfolio. Also, assume that the death and withdrawal events happen at anniversary dates. Adapting Equation (4), the fund value at each anniversary date $t = 1, 2, \dots, T$ is simulated as

$$S_t = \exp(\mu_t + \sigma B_t),$$

where $\mu_t = \ln S_0 + (r - \sigma^2/2)t$, $\mathbf{B} \sim MVN(\mathbf{0}, \Sigma)$, and $\Sigma_{ij} = \min(i, j)$. In our experiments, following the work of Gan (2013) and Gan and Lin (2015), we use parameters $S_0 = 1$, $r = 3\%$, and $\sigma = 20\%$.

As the expected VA value $E[V(\mathcal{S})]$ has no closed-form analytical formula, we first conduct a benchmark simulation experiment to provide accurate estimates of the synthetic portfolio and its constituent contracts. Specifically, we run a Monte Carlo simulation experiment with 100,000 independent sample paths for the underlying fund. The contract value and delta for each of the 100,000 VA contracts in the synthetic portfolio are then calculated based on these 100,000 sample paths. Note that, in practice, such an accurate simulation can take unbearably long when the VA contracts have complicated payoffs that are linked to multiple funds and other risk factors such as stochastic interest rate and stochastic volatility. The benchmark portfolio value in this large simulation is \$740,817,289; the standard error is about 0.4561% of the portfolio value. The benchmark portfolio delta is $-2,095,370,730$; the standard error is about 0.3293% of the portfolio delta. The negative delta means that the portfolio value, which is the insurer’s liability, decreases with the initial sub-account value. The small standard error shows that this estimate is indeed accurate and can serve as a benchmark for assessing the accuracy of other methods. We will referred to these estimates as the “true values” in subsequent discussions.

We compare and contrast 4 methods, the Monte Carlo simulation with different numbers of sample paths and three QMC methods with different PGMs with comparable number of sample paths. Both the portfolio- and the contract-level accuracies of these methods are considered. Specifically, we examine the mean absolute percentage error (MAPE) and the mean relative error (MRE), given

$$\text{MAPE} = \frac{\sum_{i=1}^{10^5} |\hat{\mu}_i - \mu_i|}{\sum_{i=1}^{10^5} \mu_i} \quad \text{and} \quad \text{MRE} = \frac{1}{10^5} \sum_{i=1}^{10^5} \frac{|\hat{\mu}_i - \mu_i|}{\mu_i}, \tag{8}$$

where $\hat{\mu}_i$ and μ_i are the estimated portfolio value (delta) and the true portfolio value (delta) of the i -th VA contract, respectively.

For the QMC methods in our experiments, the QMC senescence is generated using randomized Sobol points with a random linear scramble and random digital shift (See Sobol (1967) and Niederreiter (1992) for details). The randomization of deterministic QMC sequence allows us to conduct independent experiments to estimate the value (delta) of the same synthetic portfolio. The sample average and the standard error of MAPEs and MREs for different QMC methods can then be estimated from these repeated experiment and be compared to those for the Monte Carlo simulation. In subsequent discussions, we perform 100 independent simulation experiments for every method. Also, the number of sample paths in all subsequent experiments are powers of 2 to accommodate convenience implementation of the Brownian bridge construction of Brownian path. Table 2 summarizes the results of our numerical experiments.

Table 2: MAPEs and MREs for the market values and dollar deltas of a synthetic portfolio with 100,000 VA contracts. MAPEs and MREs are performance measures for the portfolio- and contract-level accuracies. Numbers in parentheses show the standard errors of the MAPEs and MREs, estimated from the 100 repeated experiments.

Method	Portfolio Value		Portfolio Dollar Delta	
	MAPE	MRE	MAPE	MRE
<i>Number of sample paths</i> = $1024 = 2^{10}$				
MC	3.61% (0.0255)	3.76% (0.0236)	3.27% (0.0164)	3.01% (0.0139)
QMC-CHO	0.97% (0.0050)	1.04% (0.0043)	2.27% (0.0062)	1.93% (0.0042)
QMC-BB	0.54% (0.0014)	0.52% (0.0009)	1.78% (0.0025)	1.37% (0.0016)
QMC-PCA	0.29% (0.0006)	0.32% (0.0006)	1.10% (0.0013)	0.87% (0.0010)
<i>Number of sample paths</i> = $256 = 2^8$				
QMC-CHO	2.09% (0.0119)	2.33% (0.0098)	4.51% (0.0156)	3.92% (0.0105)
QMC-BB	1.19% (0.0031)	1.19% (0.0027)	3.67% (0.0050)	2.91% (0.0037)
QMC-PCA	0.56% (0.0020)	0.61% (0.0019)	2.66% (0.0039)	2.04% (0.0025)
<i>Number of sample paths</i> = $64 = 2^6$				
QMC-CHO	5.13% (0.0291)	5.99% (0.0251)	10.10% (0.0400)	8.61% (0.0277)
QMC-BB	3.46% (0.0110)	3.45% (0.0106)	8.84% (0.0191)	6.92% (0.0104)
QMC-PCA	1.77% (0.0075)	2.05% (0.0087)	6.28% (0.0113)	4.99% (0.0061)

We first consider experiments with $1024 = 2^{10}$ sample paths, which is approximately 10 times less sample paths than the benchmark simulation. The MAPEs and MREs for this setting are summarized in the first panel of Table 2. We see that all three QMC methods produce lower MAPEs and MREs than the Monte Carlo method with the same number of sample paths, for both the portfolio value and the portfolio dollar delta. This means that the QMC estimates are more accurate at both the portfolio- and the contract-levels. We could also see that the accuracies are different among the three PGM methods: the QMC-PCA produces the most accurate estimates by concentrating its explained variability in the first few dimensions. In contrast, the Cholesky decomposition path generation method, which is the standard method in Monte Carlo, gives no particular attention to concentrating variability and so produces the least accurate estimates among the three QMC methods. The accuracy of the Brownian bridge method lies between the other two. In summary, in this example and with $2^{10} = 1024$ sample paths, QMC method using the PCA path generation method can be 3-33 times more accurate than the standard Monte Carlo method.

We perform further experiments to examine the accuracies of different QMC methods with less number of sample paths. From the second panel of Table 2, we could see obviously that the three QMC methods with 256 sample paths can produce about the same accuracy as the Monte Carlo method with 1024 sample

paths. In fact, the QMC-PCA method with 256 sample paths is more accurate than Monte Carlo with 1024 sample paths, for both portfolio value and dollar delta estimates in both MAPE and MRE. In the third panel of Table 2, we see that even with merely 64 sample paths, the QMC-PCA method can still produce more accurate portfolio value estimates than the Monte Carlo method with 1024 sample paths. This suggests an at least 16 times computational savings, which can manifest as the difference between days and hours of runtime in practical applications.

5 CONCLUDING REMARKS

In this study we propose and test QMC method with three different path generation methods in the valuation and risk management for large variable annuity (VA) portfolios. Our numerical experiments show that all three QMC methods produce more accurate estimates of large VA portfolios and their sensitivity to risk factors. The principal component analysis method showing the most promising numerical results, which matches its proven theoretical properties.

REFERENCES

- Akesson, F. 1998. "Discrete Eigenfunction Expansion of Multi-Dimensional Brownian Motion and the Ornstein-Uhlenbeck Process 1 One-Dimensional Brownian Motion". Technical report, Carnegie-Mellon University.
- Boyle, P. 1977. "Options: A Monte Carlo Approach". *Journal of Financial Economics* 4(3):323–338.
- Boyle, P., M. Broadie, and P. Glasserman. 1997. "Monte Carlo Methods for Security Pricing". *Journal of Economic Dynamics and Control* 21(8-9):1267–1321.
- Bratley, P., B. L. Fox, and H. Niederreiter. 1992. "Implementation and Tests of Low-Discrepancy Sequences". *ACM Transactions on Modeling and Computer Simulation (TOMACS)* 2(3):195–213.
- Caflich, R., W. Morokoff, and A. Owen. 1997. "Valuation of Mortgage-Backed Securities Using Brownian Bridges to Reduce Effective Dimension". *The Journal of Computational Finance* 1(1):27–46.
- Feng, B. M., Z. Tan, and J. Zheng. 2020. "Efficient Simulation Designs for Valuation of Large Variable Annuity Portfolios". *North American Actuarial Journal* 24(2):275–289.
- Gan, G. 2013. "Application of Data Clustering and Machine Learning in Variable Annuity Valuation". *Insurance: Mathematics and Economics* 53(3):795–801.
- Gan, G., and X. S. Lin. 2015. "Valuation of Large Variable Annuity Portfolios under Nested Simulation: A Functional Data Approach". *Insurance: Mathematics and Economics* 62:138–150.
- Gan, G., and X. S. Lin. 2017. "Efficient Greek Calculation of Variable Annuity Portfolios for Dynamic Hedging: A Two-Level Metamodeling Approach". *North American Actuarial Journal* 21(2):161–177.
- Gan, G., and E. A. Valdez. 2018. "Regression Modeling for the Valuation of Large Variable Annuity Portfolios". *North American Actuarial Journal* 22(1):40–54.
- Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*, Volume 53. New York, New York: Springer Science and Business Media.
- Hardy, R. M. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*, Volume 1. Hoboken, New Jersey: John Wiley and Sons.
- Hejazi, S. A., and K. R. Jackson. 2016. "A Neural Network Approach to Efficient Valuation of Large Portfolios of Variable Annuities". *Insurance: Mathematics and Economics* 70:169–181.
- Hejazi, S. A., K. R. Jackson, and G. Gan. 2017. "A Spatial Interpolation Framework for Efficient Valuation of Large Portfolios of Variable Annuities". *Quantitative Finance and Economics* 1(2):1–5.
- Johansen, R. J. 1996. "Annuity 2000 Mortality Tables". *Transactions Of Society Of Actuaries, 1995-96 Reports*.
- Joy, C., P. Boyle, and K. S. Tan. 1996. "Quasi-Monte Carlo Methods in Numerical Finance". *Management Science* 42(6):926–938.
- L'Ecuyer, P. 2004. "Quasi-Monte Carlo Methods in Finance". In *Proceedings of the 2004 Winter Simulation Conference*, edited by R. G. Ingalls, M. D. Rossetti, J. S. Smith, and B. A. Peters, 1645–1655. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.
- Liu, K. 2012. *Directional Control of Generating Brownian Path under Quasi Monte*. Ph. D. thesis, University of Waterloo.
- Liu, K., and K. S. Tan. 2020. "Real-Time Valuation of Large Variable Annuity Portfolios: A Green Mesh Approach". *North American Actuarial Journal*. Published online, Jan 2020.
- Liu, Y. 2010. *Pricing and Hedging the Guaranteed Minimum Withdrawal Benefits in Variable Annuities*. Ph.D. thesis, Department of Statistics and Actuarial Science, University of Waterloo, Waterloo, Ontario, Canada. https://uwspace.uwaterloo.ca/bitstream/handle/10012/4990/Liu_Yan.pdf?sequence=1.

- Moskowitz, B., and R. E. Caflisch. 1996. "Smoothness and Dimension Reduction in Quasi-Monte Carlo Methods". *Mathematical and Computer Modelling* 23(8-9):37–54.
- Niederreiter, H. 1992. *Random Number Generation and Quasi-Monte Carlo Methods*. Philadelphia: Society for Industrial and Applied Mathematics (SIAM).
- Owen, A. 2003. "The Dimension Distribution and Quadrature Test Functions". *Statistica Sinica* 13(1):1–17.
- Papageorgiou, A. 2000. "Fast Convergence of Quasi-Monte Carlo for a Class of Isotropic Integrals". *Mathematics of Computation* 70(233):297–307.
- Scheicher, K. 2007. "Complexity and Effective Dimension of Discrete Lévy Areas". *Journal of Complexity* 23(2):152–168.
- Secure Retirement Institute 2020. "Secure Retirement Institute Annuity Sales Estimates 2010-2019".
- Sloan, I. H., and H. Woźniakowski. 1998. "When Are Quasi-Monte Carlo Algorithms Efficient for High Dimensional Integrals?". *Journal of Complexity* 14(1):1–33.
- Sloan, I. H., and H. Woźniakowski. 2001. "Tractability of Multivariate Integration for Weighted Korobov Classes". *Journal of Complexity* 17(4):697–721.
- Sobol, I. M. 1967. "On the Distribution of Points in a Cube and the Approximate Evaluation of Integrals". *USSR Computational Mathematics and Mathematical Physics* 7(4):86–112.
- Wang, X., and K. S. Tan. 2012. "How Do Path Generation Methods Affect the Accuracy of Quasi-Monte Carlo Methods for Problems in Finance?". *Journal of Complexity* 28(2):250–277.
- Wang, X., and K. S. Tan. 2013. "Pricing and Hedging with Discontinuous Functions: Quasi-Monte Carlo Methods and Dimension Reduction". *Management Science* 59(2):376–389.
- Xu, W., Y. Chen, C. Coleman, and T. F. Coleman. 2018. "Moment Matching Machine Learning Methods for Risk Management of Large Variable Annuity Portfolios". *Journal of Economic Dynamics and Control* 87:1–20.

AUTHOR BIOGRAPHIES

BEN MINGBIN FENG is an assistant professor in actuarial science at the University of Waterloo. He earned his Ph.D. in the Department of Industrial Engineering and Management Sciences at Northwestern University. He is an Associate of the Society of Actuaries (ASA). His research interests include stochastic simulation design and analysis, optimization via simulation, nonlinear optimization, and financial and actuarial applications of simulation and optimization methodologies. His e-mail address is ben.feng@uwaterloo.ca. His website is <http://www.math.uwaterloo.ca/~mbfeng/>.

Kai Liu is an assistant professor in the School of Mathematical and Computational Sciences at the University of Prince Edward Island and the head of Machine Learning Group in the big data research lab at the University of Waterloo. He obtained all of his degrees from the University of Waterloo: Ph.D. in actuarial science (2017), M.Math in actuarial science (2012) and B.Math in actuarial science and computer science (2011). In addition, he is an Associate of the Society of Actuaries (ASA) and the Canadian Institute of Actuaries (ACIA). His research interests are actuarial science, big data analytics, financial technology, quantitative finance, and machine learning. His e-mail address is kailiu@upei.ca. His website is <https://www.upei.ca/profile/kai-liu>.