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DYNAMIC SAMPLING PROCEDURE FOR DECOMPOSABLE RANDOM NETWORKS

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ABSTRACT

This research studies the problem of node ranking in a random network. Specifically, we consider a Markov chain with several ergodic classes and unknown transition probabilities which can be estimated by sampling. The objective is to select all of the best nodes in each ergodic class. A sampling procedure is proposed to decompose the Markov chain and maximize a weighted probability of correct selection of the best nodes in each ergodic class. Numerical results demonstrate the efficiency of the proposed sampling procedure.

1 INTRODUCTION

A central problem in analyzing many social and economic networks is to rank and select the important nodes, which is referred to as the nodes ranking and selection (R&S) problem henceforth. For example, Google's search engine uses the acclaimed PageRank algorithm to list the most important web pages for each keyword (Brin and Page 1998; Page 1998); in social network like Twitter, influential members are ranked and selected for popularity recommendation (Weng et al. 2010). Other examples include venture capitalists selection (Bhat and Sims 2012) and academic paper searching (Walker et al. 2007; Jomsri et al. 2011).

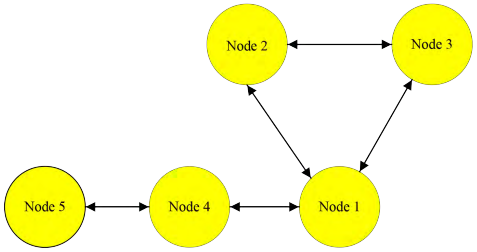
In the world wide web and social networks, nodes are linked with each other randomly with certain probabilities. The nodes R&S problem considers a Markov chain constructed from networks (see, e.g., Langville and Meyer 2011; Berkhout and Heidergott 2018a; Berkhout and Heidergott 2018b). Consider a finite node set $\mathbb{S} = \{1, \ldots, n\}$ and a Markov chain $P = [P_{ij}]_{n \times n}$ defined on $\mathbb{S}$, where P_{ij} denotes the transition probability of visiting node j from node i. For instance, a simple network and its corresponding transition probability matrix are shown in Figure 1. The Markov chain constructed from this network has a single ergodic class consisting of five nodes. The importance of the nodes is ranked by their corresponding stationary probabilities of the Markov chain. Specifically, $\pi \triangleq (\pi_1, \ldots, \pi_n)$ is the vector of the stationary probabilities, where π_i denotes the stationary probability of node i which captures the long-run proportion of visits to node i by the Markov chain. According to the following equilibrium equation:

$$\pi P = \pi, \text{ and } \sum_{i=1}^{n} \pi_i = 1, \ \pi_i > 0, \tag{1}$$

we obtain $\pi = (12/58, 3/58, 3/58, 24/58, 16/58)$ which implies that the importance ranking inside the ergodic class is

$$4 \succ 5 \succ 1 \succ 2 \sim 3,$$

where $i \succ j$ means node i is more important than node j, and $i \sim j$ means nodes i and j are equally important.



$$\mathbf{P} = \left[\begin{array}{ccc|cc} 0.798 & 0.1 & 0.1 & 0.002 & 0 \\ 0.4 & 0.35 & 0.25 & 0 & 0 \\ 0.4 & 0.25 & 0.35 & 0 & 0 \\ \hline 0.001 & 0 & 0 & 0.599 & 0.4 \\ 0 & 0 & 0 & 0.6 & 0.4 \end{array}\right]$$

Figure 1: Small network example.

A Markov chain may take a long time to transit between the nodes in different subclasses (Meyer 1989). To facilitate the spread of information among the nodes in a Markov chain, it is advisable to decompose the Markov chain and then select the most important nodes in each subclass. For example, consider a marketing targeting problem in the network described in Figure 1, where nodes 4 and 5 are most influential in the Markov chain, which appears to suggest them as the natural candidates to be targeted by advertisement. However, notice that both $P_{1,4}$ and $P_{4,1}$ are negligible and thus it usually takes a long time to transit between two subclasses, i.e., (1, 2, 3) and (4, 5). If we set $P_{1,4}$ and $P_{4,1}$ to zero and normalize the rows, the original network will be decomposed into two ergodic classes. Targeting both node 1 and node 4, the two most influential nodes in their respective subclasses, would lead to much better information spread.

In a random network, the transition probabilities are unknown and are functions of certain interaction parameters which can be estimated by sampling the interactions between the nodes. However, randomness in sampling brings in two major challenges to the nodes R&S problem. First, errors in transition probability estimation may lead to misclassification of the nodes in the decomposition process. Second, even if the nodes are decomposed into correct subclasses, errors in interaction parameter estimation will still lead to errors in stationary probability estimation of each node, which in turn results in ranking and selecting the nodes incorrectly.

R&S has been studied actively in simulation, which considers selecting the best or an optimal subset from finite alternatives (see, e.g., Chen et al. 2008; Chen and Lee 2011; Luo et al. 2015; Peng et al. 2018). However, few R&S studies are in the context of network or Markov chain. Li et al. (2018) and Li et al. (2019) propose dynamic sampling procedures to maximize the probability of correctly selecting the most important nodes in a random network, but they limit their discussion to the Markov chain with a single ergodic class and do not consider network decomposition. To fill this gap, we aim to develop a dynamic sampling procedure to simultaneously decompose the Markov chain into several classes and maximize a weighted probability of correct selection (WPCS) of the best nodes in each ergodic class.

The rest of the paper is organized as follows. In Section 2, we formulate the problem of Markov chain decomposition and node selection in a random network. Section 3 provides a decomposition algorithm for a random network with an asymptotic guarantee on the probability of correct decomposition to any pre-specified level. An efficient dynamic sampling procedure is proposed in Section 4, and numerical results are presented in Section 5. The last section concludes the paper and outlines future directions.

2 PROBLEM FORMULATION

We first provide a list of the notations frequently used in this study, and additional notations and definitions will be introduced when needed.

$\mathscr{X}$	vector of all interaction parameters between pairs of nodes;
x_{ij}	interaction parameter between nodes i and j;
P	transition probability matrix of Markov chain;
π	vector of stationary probabilities;
$\mathscr{X}^{(t)}$	posterior estimate of $\mathscr{X}$ after allocating t samples;
a_{ij}	node j of ergodic class i in Markov chain;
$a_{\langle i,j\rangle_t}$	the node with the j-th largest posterior estimate of stationary probability in ergodic class i after allocating t samples.

We consider a random network consisting of n nodes. A Markov chain of the network is constructed by the interaction strength between each pair of nodes. Specifically, the transition probabilities $P_{ij} = P_{ij}(\mathscr{X})$, $i, j = 1, \ldots, n$, are functions of a vector $\mathscr{X} \triangleq (x_{ij})_{1 \le i < j \le n}$, where each interaction parameter x_{ij} denotes the proportion of visits from node j to node i among all interactive visits between nodes i and j.

In the random network, all interaction parameters x_{ij}, $1 \le i < j \le n$, are assumed to be unknown but can be estimated by sampling. Let $X_{ij,t}, t \in \mathbb{Z}^+$ be the t-th sample for the interactions between nodes i and j, which is assumed to follow an independent and identically distributed (i.i.d.) Bernoulli distribution with unknown parameter x_{ij}, $1 \le i < j \le n$. The Bernoulli assumption is natural in many practical applications. For instance, in the PageRank (Langville and Meyer 2011; Langville and Meyer 2012), a binary variable takes 1 for the visits from web page j to i and takes 0 for the reverse direction. Suppose the prior distribution of x_{ij} is $U[0,1]$. By conjugacy (Gelman et al. 2014), the posterior distribution of x_{ij} is a Beta distribution with the posterior mean:

$$x_{ij}^{(t)} \triangleq \alpha_{ij}^{(t)} / (\alpha_{ij}^{(t)} + \beta_{ij}^{(t)}),$$

and the posterior variance:

$$(\sigma_{ij}^{(t)})^2 \triangleq \alpha_{ij}^{(t)} \beta_{ij}^{(t)} / \left[(\alpha_{ij}^{(t)} + \beta_{ij}^{(t)})^2 (\alpha_{ij}^{(t)} + \beta_{ij}^{(t)} + 1) \right],$$

where

$$\alpha_{ij}^{(t)} \triangleq 1 + \sum_{\ell=1}^{t_{ij}} X_{ij,\ell}, \quad \beta_{ij}^{(t)} \triangleq 1 + \sum_{\ell=1}^{t_{ij}} (1 - X_{ij,\ell}),$$

and t_{ij} is the number of samples allocated to estimate x_{ij} after allocating t samples where $t = \sum_{1 \le i < j \le n} t_{ij}$. With the estimates of the interaction parameters, the transition probabilities and stationary probabilities can be in turn estimated.

2.1 Network Decomposition

In this study, we adopt the Kemeny decomposition algorithm proposed by Berkhout and Heidergott (2019). The Kemeny constant is a weighted average of mean first passage times. For Markov chains with several ergodic classes, the Kemeny constant is formally defined as

$$K_P = \mathrm{tr}(D_P) + 1,$$

where D_P is the deviation matrix of a Markov chain $P = [P_{ij}]_{n \times n}$, given by

$$D_P = (I - P + \Pi_P)^{-1} - \Pi_P,$$

where I is an $n \times n$ identity matrix and $\Pi_P = [\Pi_P(i,j)]_{n \times n}$ is called the ergodic projector of P, given by

$$\Pi_P = \lim_{N \to +\infty} \frac{1}{N} \sum_{i=0}^{N-1} P^i.$$

A network modeled by a Markov chain with a small Kemeny constant has relatively good connectivity and vice versa.

In the Kemeny decomposition algorithm, the Kemeny constant is differentiated with respect to any entry of P. For each entry (i,j), the derivative of the Kemeny constant is defined by $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$, where

$$P_{\theta,ij} = (1-\theta)P + \theta R_{ij},\ \theta \in [0,1],$$

and

$$R_{ij} = P - e_i e_i^{'} P + e_i e_j^{'} = P - e_i(e_i^{'} P - e_j^{'}),$$

where e_i indicates the i-th column of the identity matrix. It follows from Theorem 1 of Berkhout and Heidergott (2019) that

$$\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0} = \mathrm{tr}\left((R_{ij}-P)(D_P)^2\right). \tag{2}$$

An entry (i,j) with a small negative value for $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$ means that increasing the transition probability from node i to node j has a large positive impact on connectivity of network. Setting P_{ij} to zero and normalizing the row afterward will decrease connectivity of the network significantly. The algorithm removes the link with the smallest derivative of the Kemeny constant until some stopping criteria hold, for example, no entry with $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0} < 0$. See more details about this algorithm in Berkhout and Heidergott (2019).

2.2 Node Selection

After decomposition, suppose n nodes are decomposed into k ergodic classes. Specifically, nodes $a_{11},\ldots,a_{1m_1}$ belong to ergodic class 1, ... , and nodes $a_{k1},\ldots,a_{km_k}$ belong to ergodic class k, where $\sum_{i=1}^{k} m_i = n$. In each ergodic class i, its transition probability matrix is denoted by P_i and the stationary probability for node a_{ij} is denoted by $\pi_{a_{ij}}$. Without loss of generality, suppose that $\pi_{a_{11}} \geq \cdots \geq \pi_{a_{1m_1}}$, ... , and $\pi_{a_{k1}} \geq \cdots \geq \pi_{a_{km_k}}$. Our objective is to select the best nodes of each ergodic class:

$$\Theta_k \triangleq \{a_{11},\ldots,a_{k1}\}.$$

Suppose the number of samples is fixed. Given the information of s allocated samples, we use a posterior estimate of $\mathscr{X}$, namely $\mathscr{X}^{(s)} \triangleq (x_{ij}^{(s)})_{1\leq i<j\leq n}$, to estimate the transition probabilities and stationary probabilities. For each ergodic class i, a posterior estimate of transition probability matrix P_i is $P_i(\mathscr{X}^{(s)})$. Solving equilibrium equation (1) by plugging in $P_i(\mathscr{X}^{(s)})$ yields a posterior estimate of the vector of the stationary probabilities in ergodic class i:

$$\Pi_i(\mathscr{X}^{(s)}) = \left(\pi_{a_{i1}}(\mathscr{X}^{(s)}),\ldots,\pi_{a_{im_i}}(\mathscr{X}^{(s)})\right).$$

Our final selection is to pick the nodes with the largest posterior estimates of the stationary probabilities in each ergodic class, i.e.,

$$\widehat{\Theta}_k^{(s)} \triangleq \left\{a_{\langle 1,1\rangle_s},\ldots,a_{\langle k,1\rangle_s}\right\},$$

where $a_{\langle i,\cdot\rangle_s}$ are the ranking indices in ergodic class i such that

$$\pi_{a_{\langle i,1\rangle_s}}(\mathscr{X}^{(s)}) \geq \cdots \geq \pi_{a_{\langle i,m_i\rangle_s}}(\mathscr{X}^{(s)}).$$

The research problem of this study is to sequentially allocate each sample based on available information collected throughout previous sampling at each step to estimate the interaction parameters between different pairs of nodes for efficiently selecting the best nodes of each ergodic class. We measure the statistical efficiency of a sampling procedure by a weighted probability of correct selection (WPCS) of the best nodes in each ergodic class defined as follows:

$$\sum_{i=1}^{k} m_i \Pr\left(a_{\langle i,1\rangle_s} = a_{i1}\right)/n.$$

3 DECOMPOSITION GUARANTEE

Since randomness in sampling interactions leads to estimation errors in transition probabilities and stationary probabilities, the derivative estimate of the Kemeny constant $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$ in equation (2) also contains errors. Note that the distribution of $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$ does not have a closed form in Bayesian learning, so we use the first-order Taylor expansion to approximate the posterior distribution of $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$, i.e.,

$$\frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X})\Big|_{\theta=0} \approx \frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X}^{(t)})\Big|_{\theta=0} + \sum_{1\le k<\ell\le n}\left[\frac{\partial}{\partial x_{k\ell}}\left(\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}\right)\Bigg|_{\mathscr{X}=\mathscr{X}^{(t)}}\left(x_{k\ell}-x_{k\ell}^{(t)}\right)\right].$$

We adapt the Kemeny decomposition algorithm to the random network by establishing an asymptotic guarantee on the probability of correct decomposition to any pre-specified level $(1-\alpha)$. Specifically, let $\mathbb{E}^{(t)} \triangleq \left\{(i,j)|P_{ij}(\mathscr{X}^{(t)})>0, i,j\in\mathbb{S}\right\}$ denote the set of links and (i^*,j^*) denote the link with the smallest derivative estimates of the Kemeny constant, i.e.,

$$(i^*,j^*) \triangleq \underset{(i,j)\in\mathbb{E}^{(t)}}{\arg\min}\ \frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X}^{(t)})\Big|_{\theta=0}.$$

Link (i^*,j^*) will be removed if and only if the posterior probability of correct removal is reached:

$$\Pr\left(\bigcap_{(i,j)\in\mathbb{E}^{(t)}/(i^*,j^*)}\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}\Big|_{\theta=0} < \frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}\right)\Bigg|\mathscr{E}_t\right) \ge 1-\alpha, \tag{3}$$

where $\mathscr{E}_t$ is the information collected throughout the t-th sample. However, calculating the posterior probability of correct removal in (3) still suffers from heavy computational burden. To address this difficulty, we apply the Bonferroni inequality:

$$\begin{aligned}&\Pr\left(\bigcap_{(i,j)\in\mathbb{E}^{(t)}/(i^*,j^*)}\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}\Big|_{\theta=0} < \frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}\right)\Bigg|\mathscr{E}_t\right)\\ \ge\ & 1-\sum_{(i,j)\in\mathbb{E}^{(t)}/(i^*,j^*)}\Pr\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}\Big|_{\theta=0} > \frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}\Bigg|\mathscr{E}_t\right).\end{aligned}$$

Therefore,

$$\sum_{(i,j)\in\mathbb{E}^{(t)}/(i^*,j^*)}\Pr\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}\Big|_{\theta=0} - \frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0} > 0\Bigg|\mathscr{E}_t\right) \le \alpha$$

can guarantee the posterior probability of correct removal to level $(1-\alpha)$. However, the Bonferroni inequality may only offer a very loose lower bound, which in turn results in requiring much more samples than necessary for reaching the guaranteed probability level.

Notice that normal distribution family is closed under linear combinations. Based on the asymptotic normality of the Beta distribution in Bayesian estimation established in Li et al. (2019), the following asymptotic result can be expected:

$$\lim_{t\to+\infty}\left|\Pr\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}\Big|_{\theta=0}-\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}>0\,\middle|\,\mathscr{E}_t\right)-\Phi\left(\frac{\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}(\mathscr{X}^{(t)})\Big|_{\theta=0}-\frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X}^{(t)})\Big|_{\theta=0}}{\tau^{(t)}_{i^*j^*ij}}\right)\right|=0,$$

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal distribution and

$$(\tau^{(t)}_{i^*j^*ij})^2 \triangleq \sum_{1\le k<\ell\le n}\left[\left(\frac{\partial}{\partial x_{k\ell}}\left(\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}-\frac{d}{d\theta}K_{P_{\theta,ij}}\right)\Big|_{\theta=0}\right)\Big|_{\mathscr{X}=\mathscr{X}^{(t)}}\right)^2(\sigma^{(t)}_{k\ell})^2\right].$$

To enhance computational efficiency, we replace the guarantee condition (3) for decomposition by

$$\sum_{(i,j)\in\mathbb{E}^{(t)}/(i^*,j^*)}\Phi\left(\frac{\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}(\mathscr{X}^{(t)})\Big|_{\theta=0}-\frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X}^{(t)})\Big|_{\theta=0}}{\sqrt{\sum_{1\le k<\ell\le n}\left[\left(\frac{\partial}{\partial x_{k\ell}}\left(\left(\frac{d}{d\theta}K_{P_{\theta,i^*j^*}}-\frac{d}{d\theta}K_{P_{\theta,ij}}\right)\Big|_{\theta=0}\right)\Big|_{\mathscr{X}=\mathscr{X}^{(t)}}\right)^2(\sigma^{(t)}_{k\ell})^2\right]}}\right)\le\alpha. \quad (4)$$

The derivative of $\frac{d}{d\theta}K_{P_{\theta,ij}}\Big|_{\theta=0}$ with respect to $x_{k\ell}$ can be calculated as follows:

$$\begin{aligned}
&\frac{\partial}{\partial x_{k\ell}}\left(\frac{d}{d\theta}K_{P_{\theta,ij}}(\mathscr{X})\Big|_{\theta=0}\right)\\
=\;&\frac{\partial}{\partial x_{k\ell}}\mathrm{tr}\left((R_{ij}(\mathscr{X})-P(\mathscr{X}))(D_P(\mathscr{X}))^2\right)\\
=\;&\mathrm{tr}\left(\frac{\partial}{\partial x_{k\ell}}\left((R_{ij}(\mathscr{X})-P(\mathscr{X}))(D_P(\mathscr{X}))^2\right)\right)\\
=\;&\mathrm{tr}\left(\left(\frac{\partial}{\partial x_{k\ell}}(R_{ij}(\mathscr{X})-P(\mathscr{X}))\right)(D_P(\mathscr{X}))^2+(R_{ij}(\mathscr{X})-P(\mathscr{X}))\frac{\partial}{\partial x_{k\ell}}(D_P(\mathscr{X}))^2\right)\\
=\;&\mathrm{tr}\left(-e_ie_i'\frac{\partial P(\mathscr{X})}{\partial x_{k\ell}}(D_P(\mathscr{X}))^2+(R_{ij}(\mathscr{X})-P(\mathscr{X}))\left[\left(\frac{\partial D_P(\mathscr{X})}{\partial x_{k\ell}}\right)D_P(\mathscr{X})+D_P(\mathscr{X})\left(\frac{\partial D_P(\mathscr{X})}{\partial x_{k\ell}}\right)\right]\right),
\end{aligned}$$

where $\frac{\partial P(\mathscr{X})}{\partial x_{k\ell}}\triangleq\left[\frac{\partial P_{ij}(\mathscr{X})}{\partial x_{k\ell}}\right]_{n\times n}$ is the derivative of the transition matrix P with respect to $x_{k\ell}$. Note that $D_P=(I-P+\Pi_P)^{-1}-\Pi_P$, and then

$$\begin{aligned}
&\frac{\partial D_P(\mathscr{X})}{\partial x_{k\ell}}\\
=\;&-(I-P(\mathscr{X})+\Pi_P(\mathscr{X}))^{-1}\left(\frac{\partial}{\partial x_{k\ell}}(I-P(\mathscr{X})+\Pi_P(\mathscr{X}))\right)(I-P(\mathscr{X})+\Pi_P(\mathscr{X}))^{-1}-\frac{\partial\Pi_P(\mathscr{X})}{\partial x_{k\ell}}\\
=\;&(I-P(\mathscr{X})+\Pi_P(\mathscr{X}))^{-1}\left(\frac{\partial P(\mathscr{X})}{\partial x_{k\ell}}-\frac{\partial\Pi_P(\mathscr{X})}{\partial x_{k\ell}}\right)(I-P(\mathscr{X})+\Pi_P(\mathscr{X}))^{-1}-\frac{\partial\Pi_P(\mathscr{X})}{\partial x_{k\ell}}.
\end{aligned}$$

In each ergodic class i, the derivative of the stationary distribution vector denoted by

$$\frac{\partial \Pi_i(\mathscr{X})}{\partial x_{k\ell}} \triangleq \left(\frac{\partial \pi_{a_{i1}}(\mathscr{X})}{\partial x_{k\ell}}, \ldots, \frac{\partial \pi_{a_{im_i}}(\mathscr{X})}{\partial x_{k\ell}} \right)$$

is a solution of the following set of equations (Li et al. 2019):

$$\begin{cases} \dfrac{\partial \Pi_i(\mathscr{X})}{\partial x_{k\ell}} [I - P_i(\mathscr{X})] &= \Pi_i(\mathscr{X}) \dfrac{\partial P_i(\mathscr{X})}{\partial x_{k\ell}}, \\ \displaystyle\sum_{j=1}^{m_i} \frac{\partial \pi_{a_{ij}}(\mathscr{X})}{\partial x_{k\ell}} &= 0 \, . \end{cases}$$

Then $\dfrac{\partial \Pi_P(\mathscr{X})}{\partial x_{k\ell}}$ can be obtained by merging the derivatives of stationary probabilities in all ergodic classes.

4 DYNAMIC SAMPLING PROCEDURE

In this section, we propose a dynamic sample allocation procedure for multi-class Markov chain (DAMM), which simultaneously decomposes the Markov chain into several ergodic classes and maximizes the WPCS. Specifically, at each step t, the Markov chain of n nodes consists of k ergodic classes, and let $\mathbb{S}_i,\ i = 1, \ldots, k$, be the set of nodes in ergodic class i, given by

$$\mathbb{S}_i \triangleq \{a_{i1}, \ldots, a_{im_i}\}.$$

The next replication is allocated by

$$\widehat{A}_{t+1}(\mathscr{E}_t) \triangleq \underset{1 \le i < j \le n}{\arg\max}\ \widehat{V}_t(\mathscr{E}_t; (i,j)), \tag{5}$$

where

$$\begin{aligned} &\widehat{V}_t(\mathscr{E}_t; (i,j)) \\ \triangleq\ & \sum_{\ell=1}^{k} \frac{m_\ell}{n} \min_{\omega \in \{2,\ldots,m_\ell\}} \frac{\left(\pi_{a_{\langle \ell,1 \rangle_t}}(\mathscr{X}^{(t)}) - \pi_{a_{\langle \ell,\omega \rangle_t}}(\mathscr{X}^{(t)}) + \varepsilon \right)^2}{\displaystyle\sum_{r,q \in \mathbb{S}_\ell} \left(\frac{\partial \left(\pi_{a_{\langle \ell,1 \rangle_t}}(\mathscr{X}) - \pi_{a_{\langle \ell,\omega \rangle_t}}(\mathscr{X}) \right)}{\partial x_{rq}} \bigg|_{\mathscr{X} = \mathscr{X}^{(t)}} \right)^2 (\sigma_{rq}^{(t)})^2_{(i,j)}}, \end{aligned} \tag{6}$$

and

$$(\sigma_{rq}^{(t)})^2_{(i,j)} = \begin{cases} \dfrac{\alpha_{rq}^{(t)} \beta_{rq}^{(t)}}{(\alpha_{rq}^{(t)} + \beta_{rq}^{(t)})^2 (\alpha_{rq}^{(t)} + \beta_{rq}^{(t)} + 2)}, \text{when } (r,q) = (i,j); \\ \dfrac{\alpha_{rq}^{(t)} \beta_{rq}^{(t)}}{(\alpha_{rq}^{(t)} + \beta_{rq}^{(t)})^2 (\alpha_{rq}^{(t)} + \beta_{rq}^{(t)} + 1)}, \text{when } (r,q) \neq (i,j). \end{cases}$$

The posterior probability of correct selection in each ergodic class is an integral of the multivariate standard normal density. The approximation in deriving equation (6) relies on the exponential decay of the density of the normal distribution. See more details of this derivation in Peng et al. (2018). Also, we introduce a small positive constant ε to guarantee the consistency of the proposed sampling procedure. See more details about the constant ε in Li et al. (2019). After sampling at each step, the information set $\mathscr{E}_t$ is updated. Then the condition (4) is checked for correctly removing the link with the smallest derivative of the Kemeny constant. The corresponding x_{ij} for the removed link will not be sampled or estimated

any more. The adapted Kemeny decomposition algorithm is conducted until no link is allowed to be removed or no link with negative derivatives of the Kemeny constant. Ergodic classes of the Markov chain thereupon are updated. The sampling procedure continues until the entire sampling budget s is exhausted. The implementation of DAMM follows Algorithm 1.

Algorithm 1: Dynamic Sample Allocation Procedure for Multi-class Markov Chain

```
The total number of sampling budget is s;
t ← 1;
Initialize the number of ergodic classes k as 1, and let class 𝕊_1 be the set of all nodes;
while t ≤ s do
    Allocate a sample to estimate x_ij determined by equation (5), and then t ← t + 1;
    Update the posterior mean and variance of x_ij;
    while links with negative derivatives of the Kemeny constant exist and the condition (4) holds do
        Remove the link with the smallest derivative estimates of the Kemeny constant.
    end
    Update k and the set of nodes in each class.
end
return the best nodes in each ergodic class Θ̂_k^(s).
```

DAMM extends the DAM procedure proposed in Li et al. (2019) by considering multiple ergodic classes in a Markov chain. Similar to that in Peng et al. (2018), the sampling procedure is derived in a stochastic control framework. To avoid curse-of-dimensionality, approximate dynamic programming (ADP) is adopted. To be specific, $\widehat{V}_t(\mathscr{E}_t;(i,j))$ in equation (6) is a value function approximation (VFA) of a one-step-ahead posterior WPCS by allocating the $(t+1)$-th sample to a pair (i,j),

$$\sum_{\ell=1}^{k}\frac{m_\ell}{n}\Pr\left(\pi_{a_{\langle\ell,1\rangle_{t+1}}}>\pi_{a_{\langle\ell,\omega\rangle_{t+1}}},\forall\omega\in\{2,\ldots,m_\ell\}\middle|\mathscr{E}_t\right)\Bigg|_{(i,j)}.$$

The VFA can be considered as a weighted average of the signal-noise ratio for identifying the best node in each ergodic class. Small signal-noise ratio implies high difficulty in finding out the best node from the posterior information. The DAMM sequentially allocates each sample to estimate the interaction parameter to increase the weighted signal-noise ratio. It uses not only the posterior means and variances of interaction parameters between different nodes, but also the sensitivities of the stationary probabilities with respect to each interaction parameter. Stronger sensitivity is an indication that the corresponding posterior variance of the interaction parameter has a more significant impact on the stationary probability estimation. Intuitively, we should allocate more samples to estimate the interaction parameters that have large posterior variances and whose estimation errors can cause large errors in identifying the best node in each ergodic class.

5 NUMERICAL RESULTS

In the numerical experiments, we test the performance of different sampling procedures for Markov chain decomposition and node selection in random networks. The proposed DAMM is compared with the equal allocation (EA) and DAM in Li et al. (2019). Specifically, EA allocates one sample for each x_{ij} under consideration and conducts the same decomposition algorithm in DAMM to eliminate those x_{ij} outside ergodic classes, iteratively; DAM allocates samples according to the following rules:

$$\widehat{A}_{t+1}(\mathscr{E}_t)\triangleq\underset{1\le i<j\le n}{\arg\max}\ V_t(\mathscr{E}_t;(i,j)),$$

where

$$V_t(\mathscr{E}_t;(i,j)) \triangleq \min_{k\in\{2,\ldots,n\}} \frac{\left(\pi_{a_{\langle 1,1\rangle_t}}(\mathscr{X}^{(t)}) - \pi_{a_{\langle 1,k\rangle_t}}(\mathscr{X}^{(t)}) + \varepsilon\right)^2}{\sum_{1\le r<q\le n}\left(\left.\frac{\partial\left(\pi_{a_{\langle 1,1\rangle_t}}(\mathscr{X}) - \pi_{a_{\langle 1,k\rangle_t}}(\mathscr{X})\right)}{\partial x_{rq}}\right|_{\mathscr{X}=\mathscr{X}^{(t)}}\right)^2 (\sigma_{rq}^{(t)})^2_{(i,j)}}.$$

Note that the DAM only applies to the Markov chain with a single ergodic class, unlike DAMM and EA, DAM does not allow the Markov chain to be decomposed as sampling proceeds. Therefore, in the DAM, the same decomposition algorithm is conducted after all sampling budget is allocated. In all numerical examples, the statistical efficiency of the sampling procedures is measured by the WPCS estimated by 10,000 independent experiments. The WPCS is reported as a function of the sampling budget in each experiment.

5.1 Example 1: Courtois Matrix

The Courtois matrix is a well-known transition matrix that is often used to represent a nearly decomposable Markov chain (Stewart 1994):

$$P = \left[\begin{array}{cccc|cc|cc} 0.85 & 0 & 0.149 & 0 & 0.0009 & 0 & 0.0001 & 0 \\ 0.1 & 0.65 & 0.249 & 0.0009 & 0 & 0.00005 & 0 & 0.00005 \\ 0.1 & 0.8 & 0.0996 & 0 & 0.0003 & 0 & 0.0001 & 0 \\ \hline 0 & 0.0004 & 0 & 0.7 & 0.2995 & 0 & 0.0001 & 0 \\ 0.0005 & 0 & 0.0004 & 0.399 & 0.6 & 0.00005 & 0 & 0.00005 \\ \hline 0 & 0.00005 & 0 & 0 & 0.00005 & 0.6 & 0.2499 & 0.15 \\ 0.00003 & 0 & 0.00003 & 0.00004 & 0 & 0.1 & 0.8 & 0.0999 \\ 0 & 0.00005 & 0 & 0 & 0.00005 & 0.1999 & 0.25 & 0.55 \end{array}\right].$$

As in Google's PageRank (Langville and Meyer 2011), the Courtois matrix is constructed by

$$P_{ij} \triangleq c_{ij}x_{ji},\ 1 \le i,j \le n;$$
$$P_{ii} \triangleq 1 - \sum_{j\neq i} P_{ij},\ 1 \le i \le n,$$

where c_{ij} is a given coefficient of parameter x_{ji}. Suppose the true value of each interaction parameter is

$$x_{ij} = 0.5 + 0.03 \times (j-i),\ 1 \le i < j \le 8.$$

As the assumption in Section 2, the samples of the interaction parameter x_{ij} are generated i.i.d. from a Bernoulli distribution with parameter x_{ij}.

In Figure 2, we can see that DAMM and DAM perform better than EA, which could be attributed to the reason that EA utilizes no sample information while the other two sampling procedures utilize the information in the posterior means, variances, and sensitivities. In order to attain WPCS = 85%, DAMM needs less than 840 samples, whereas EA and DAM require more than 1200 samples. That is to say DAMM reduces the sampling budget by more than 30%. Compared with DAM, the performance enhancement of DAMM could be attributed to dynamic removal of links in the sequential sampling process, even when there is only a single ergodic class in the Markov chain. Since the decomposition algorithm in DAM is conducted after all samples are exhausted, DAM may waste some samples on estimating the interaction parameters of the links which have been removed in the sequential sampling process of DAMM. Another observation is the WPCS of the DAMM grows at a slow rate when sampling budget is around 500 and 1100, which may be caused by the increase of the number of ergodic classes.

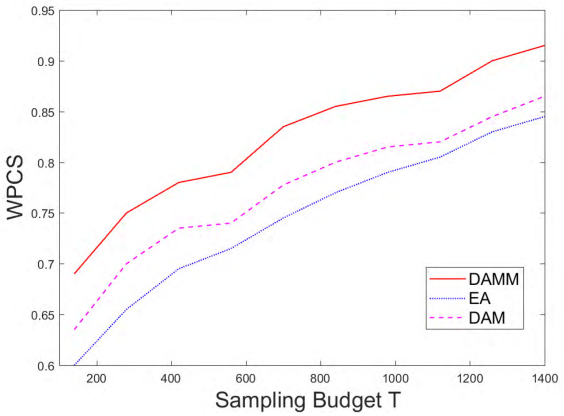


Figure 2: WPCS of the three sampling procedures in Example 1.

5.2 Example 2: Website Network

In this example, we test the performance of DAMM in a real data set from the Sogou Labs (http://www.sogou.com/labs/resource/t-link.php). The data set includes a mapping table from URL to document ID and a list of hyperlink relationship of the documents. Based on the data set, we set up two 10-node networks which do not need further decomposition. Then we randomly add 20 links between two networks, and assign a small coefficient c_{ij} (denoted in Example 1) to each added link. The true value of each interaction parameter x_{ij} inside each network is estimated from the data set, while the true value of each added x_{ij} is independently drawn from a uniform distribution $U[0,1]$.

Similar to Example 1, DAMM remains as the most efficient sampling procedure among the three, and DAM is slightly better than EA. In Figure 3, we can see the WPCS of the DAMM grows at a much faster rate than those of the EA and DAM. In order to attain WPCS = 80%, DAMM consumes less than 2500 samples, while both EA and DAM require more than 4000 samples. In addition, it is observed that the gap between the WPCS of the DAMM and those of the EA and DAM widens as the sampling budget increases.

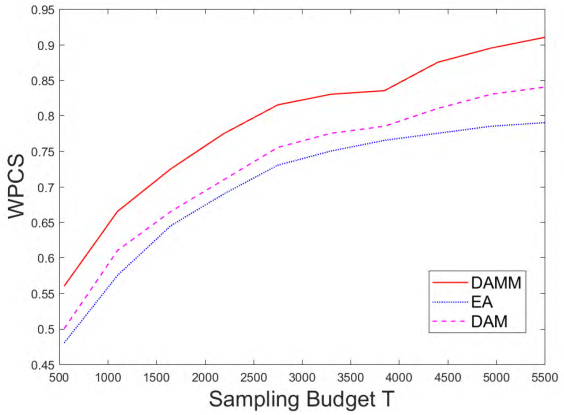


Figure 3: WPCS of the three sampling procedures in Example 2.

6 CONCLUSION

This paper deals with a sample allocation problem to optimize node selection in decomposable random networks. We propose an efficient sampling procedure named DAMM, which simultaneously maximizes the WPCS and guarantees a probability of correct decomposition to any pre-specified level. Numerical

experiments demonstrate that DAMM is significantly more efficient than the other tested sampling procedures. Future research includes the asymptotic analysis for the sampling ratio of the sequential sampling procedure for node selection in random network (Peng and Fu 2017). Considering different final rewards such as expected opportunity cost could also be an interesting future work (Gao and Chen 2015).

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REFERENCES

Berkhout, J., and B. F. Heidergott. 2018a. "Efficient Updating of Node Importance in Dynamic Real-Life Networks". *International Federation of Automatic Control-PapersOnLine* 51(7):64–69.

Berkhout, J., and B. F. Heidergott. 2018b. "Ranking Nodes in General Networks: a Markov Multi-chain Approach". *Discrete Event Dynamic Systems* 28(1):3–33.

Berkhout, J., and B. F. Heidergott. 2019. "Analysis of Markov Influence Graphs". *Operations Research* 67(3):892–904.

Bhat, H. S., and B. Sims. 2012. "InvestorRank and an Inverse Problem for PageRank". *Electronic Theses & Dissertations.* https://escholarship.org/uc/item/2qw4b90x, accessed 5th September.

Brin, S., and L. Page. 1998. "The Anatomy of a Large-scale Hypertextual Web Search Engine". *Computer Networks* 30(1-7):107–117.

Chen, C.-H., D. He, M. C. Fu, and L. H. Lee. 2008. "Efficient Simulation Budget Allocation for Selecting an Optimal Subset". *INFORMS Journal on Computing* 20(4):579–595.

Chen, C.-H., and L. H. Lee. 2011. *Stochastic Simulation Optimization: an Optimal Computing Budget Allocation*, Volume 1. Singapore: World Scientific.

Gao, S., and W. Chen. 2015. "Efficient Subset Selection for the Expected Opportunity Cost". *Automatica* 59(C):19–26.

Gelman, A., J. B. Carlin, H. S. Stern, D. B. Dunson, A. Vehtari, and D. B. Rubin. 2014. *Bayesian Data Analysis*. New York: Chemical Rubber Company Press.

Jomsri, P., S. Sanguansintukul, and W. Choochaiwattana. 2011. "CiteRank: Combination Similarity and Static Ranking with Research Paper Searching". *International Journal of Internet Technology and Secured Transactions* 3(2):161–177.

Langville, A. N., and C. D. Meyer. 2011. *Google's PageRank and Beyond: The Science of Search Engine Rankings*. Princeton: Princeton University Press.

Langville, A. N., and C. D. Meyer. 2012. *Who's #1?: The Science of Rating and Ranking*. Princeton: Princeton University Press.

Li, H., X. Xu, Y. Peng, and C.-H. Chen. 2018. "Efficient Sampling Procedure for Selecting the Largest Stationary Probability of a Markov Chain". In *2018 IEEE* 14th *International Conference on Automation Science and Engineering*, 899–905. Piscataway, New Jersey: Institute of Electrical and Electronics Engineers, Inc.

Li, H., X. Xu, Y. Peng, and C.-H. Chen. 2019. "Efficient Sampling for Selecting Important Nodes in Random Network". *arXiv:1901.03466*.

Luo, J., L. J. Hong, B. L. Nelson, and Y. Wu. 2015. "Fully Sequential Procedures for Large-Scale Ranking-and-Selection Problems in Parallel Computing Environments". *Operations Research* 63(5):1177–1194.

Meyer, C. D. 1989. "Stochastic Complementation, Uncoupling Markov Chains, and the Theory of Nearly Reducible Systems". *Society for Industrial and Applied Mathematics Review* 31(2):240–272.

Page, L. 1998. "The PageRank Citation Ranking : Bringing Order to the Web". *Stanford Digital Libraries Working Paper* 9(1):1–14.

Peng, Y., E. K. Chong, C.-H. Chen, and M. C. Fu. 2018. "Ranking and Selection as Stochastic Control". *IEEE Transactions on Automatic Control* 63(8):2359–2373.

Peng, Y., and M. C. Fu. 2017. "Myopic Allocation Policy with Asymptotically Optimal Sampling Rate". *IEEE Transactions on Automatic Control* 62(4):2041–2047.

Stewart, W. J. 1994. *Introduction to the Numerical Solution of Markov Chains*. Princeton: Princeton University Press.

Walker, D., H. Xie, K.-K. Yan, and S. Maslov. 2007. "Ranking Scientific Publications Using a Model of Network Traffic". *Journal of Statistical Mechanics: Theory and Experiment*:P06010.

Weng, J., E. P. Lim, J. Jiang, and Q. He. 2010. "TwitterRank: Finding Topic-sensitive Influential Twitterers". In *Proceedings of the third ACM international conference on Web search and data mining*, 261–270. New York: Association for Computing Machinery.

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