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HISTORY OF RANDOM VARIATE GENERATION

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ABSTRACT

Random variate generation is a fundamental aspect of simulation modeling and analysis. The objective of random variate generation is to produce observations that have the stochastic properties of a given random variable. To this end, methods and algorithms have been developed to generate random variates that are accurate (representative of the target distribution) and computationally efficient. This paper presents a history of random variate generation including distribution sampling methods used prior to the introduction of digital computers, as well as the evolution of random variate generators for continuous and discrete distributions and stochastic point processes.

1 INTRODUCTION

Given an experiment with a set of possible outcomes, a random variable is defined as a measurable function from the set of possible outcomes to a measurable space. Random variate generation is the process of producing observations (realizations) that have the stochastic properties (distribution) of a given random variable. The ability to produce stochastic simulation models for the purpose of analysis and decision making relies heavily on the use of high fidelity random variate generators. As such, the topic of random variate generation appears in many papers throughout the history of the Winter Simulation Conference dating back to the Second Conference on the Applications of Simulation in 1968 (*e.g.*, Goldberg and Mittman 1968) through today.

In the context of simulation, random variate generation (RVG) is not a stand alone concept. In particular, simulation input modeling which focuses on the design, development, and selection of appropriate probability distributions to represent system behavior is necessary to enable the generation of representative distribution values. In addition, the process of random number generation (generating uniformly distributed random number on the interval $(0,1)$) provides the fundamental *randomness* component of computer based random variate generators. As such, the history of random variate generation is heavily intertwined with the history of input modeling and the history of random number generation. For details on these topics, refer to the 2017 WSC History Track papers: *History of Input Modeling* by Russell Cheng (2017) and *History of Uniform Random Number Generation* by Pierre L'Ecuyer (2017).

The primary focus of this paper is the history of random variate generation using digital computers. However, it is important to note that generation of random variates using physical means has been used for thousands of years. Common examples include generation of discrete random variates: *casting lots* which is referenced in the Bible in Leviticus 16:8 (New King James Version) and takes place circa 1445 B.C.; urn experiments involving drawing balls of various colors from an urn; selecting lottery numbers from balls mixed in a container; flipping a coin; drawing straws; and rock–paper–scissors; among others.

Prior to the advent of computers, distribution sampling using physical means and random variate tables associated with various probability distributions were used to conduct experiments. A detailed summary of distribution sampling methods from the early 1900s through the 1940s is presented in Teichroew (1965).

The design, development, and implementation of random variate generators using digital computers first appears in the literature in the 1950's. John von Neumann (1951) authored a seminal paper introducing a set of random variate generation techniques that form the basis of many techniques in use today. The development of other algorithms soon followed including Butler (1956) and Muller (1959), among others.

Major developments in random variate generation theory and algorithm design were made between 1960 and the early 1980s, particularly in the area of simulation. Through the 1990s until the present time, random variate generation continues to be an important topic in the field of simulation. As the need to tackle increasingly complex problems grows, the ability to efficiently and accurately generate large samples from new and complex probability distributions is critical.

The remainder of this paper is organized as follows. In section 2, the concept and methods of distribution sampling (the predecessor to RVG) are discussed. The objectives of random variate generation are presented in section 3. Foundational concepts of random variate generation are presented in section 4. In section 5, the evolution of continuous and discrete random variate generators are covered. Section 6 describes methods for generating stochastic point process. Finally, concluding remarks are presented in section 7.

2 DISTRIBUTION SAMPLING (RVG PRIOR TO 1950)

Distribution sampling, or empirical sampling, can be viewed as the predecessor to modern random variate generation and was used by statisticians for many years before the development and use of RVG techniques on digital computers — prior to 1950 (Teichroew 1965). The computation of distribution functions is a fundamental problem in the theory of statistics. Namely, for a random variable, X having density function $f(x)$ ("density" is used here to apply to discrete, continuous, and mixed distributions), we want to compute

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

to determine $F(\cdot)$ as a function of x, or to determine x as a function of $F(x)$. Furthermore, given a random sample of size n, $\{X_1, X_2, \ldots X_n\}$, one may want to compute the distribution function, $H(s)$, where s is a function of the observations, such that

$$H(s_0) = \Pr\{s(X_1, X_2, \ldots, X_n) \leq s_0\} = \int \cdots \int f(x_1) f(x_2) \cdots f(x_n) dx_1 \cdots dx_n.$$

In the case in which s is simple enough, an analytical solution can be obtained. In the case, where s is a complex distribution sampling was often conducted to estimate the probabilities or to create tables for the values of the distribution function.

In 1899, William S. Gosset was employed as a brewer at Messrs. Arthur Guinness, Son and Co., Ltd. where he studied the brewing process to relate quantities of raw material of beer (barley, hops, etc.) and the conditions of production with the quality of the finished product. In doing so, his research led to the need to estimate experimental error corresponding to sample size. In 1908, Gosset published two papers under the name "Student" where he describes a sampling experiment which led to the formulation of what we refer to as the *Student-t* distribution (Pearson 1939).

In particular, Gosset (Student 1908b, Student 1908a) conducted distribution sampling experiments in which he used the height measurement of the left middle finger of 3,000 criminals from a study by Macdonell (1902). The measurements were written on individual pieces of cardboard, thoroughly shuffled and drawn, giving the measurements in random order. Consecutive groups of four measurements gave 750 samples for which the mean, standard deviation, and correlation were calculated, from which distribution curves, tables, and formulations were derived. This use of distribution sampling was unique in that it was done to enable the derivation of the mathematical formulation. E.S. Pearson (1939) states:

> *As far as I know this was the first instance in statistical research of the random sampling experiment which since has become a common and useful feature in a large number of investigations where precise analysis has failed.*

Following the lead of Gosset, other researchers conducted sampling experiments using natural or physical means to verify theoretical results (Teichroew 1965) including:

- The distribution of bacilli (bacteria) counts in a medical experiment (20,000 samples) (Greenwood and White 1910).
- Experiments to test the χ^2 distribution for goodness of fit by (a) throwing dice and tossing counters onto a spinning tray with compartments (Yule 1922), and (b) tossing coins (Brownlee 1924).
- Pearson utilizes distribution sampling to study Bayes' Theorem (*e.g.*, number of men passing a street corner carrying an umbrella; taxicabs on London streets with registration letters LX; proportion of male and female births; among others (Pearson 1925)).
- Tippett conducted large distribution sampling experiments to examine the extreme values of the normal distribution (Tippett 1925).

Tippett's (1925) experiments led to the publication of tables of random digits. In particular, Tippett (1927) produced a table of over 40,000 random digits. Tables of random digits from other researchers followed.

Nair (1938) points out that sets of random digits could be preceded by a decimal point to produce random numbers uniformly distributed on $(0,1)$. In addition, these random numbers, u, could be transformed to random variates having density $f(x)$ by what he referred to a the "probability integral transformation,"

$$u = F(x) = \int_{-\infty}^{x} f(t)dt$$

which can closely approximate a continuous distribution. Other researchers followed with additional methods for translating random digits to samples from distribution functions. As punch-card machines became available, random digit/random numbers were placed on punch-cards which reduced the computational labor involved in the distribution sampling process (Teichroew 1965).

As digital computers were introduced, researcher began to think about how random variates could be effectively utilized in computer based experiments. In particular, von Neumann (1951) suggests one alternative could be a physical device to produce a series of random digits that could be read into the computer. This method would result in the complexity of designing such a device as well as the development of methods that would be required for verification and validation. In addition, von Neumann (1951) suggests using mathematical formulations (or "cooking recipes") to generate random digits/random numbers and then proposes some "tricks" for producing random variates representing samples from other probability distribution. These latter methods and their developments are the focus of the remainder of this paper.

3 OBJECTIVES OF RANDOM VARIATE GENERATION

The objective of random variate generation is to produce sample observations that have the stochastic properties of a given random variable, X, having distribution function

$$F(x) = \Pr(X \leq x) \quad -\infty < x < \infty$$

The development of the theory/concepts surrounding random variate generation via computer algorithms is based on the following two key assumptions (Devroye 1986a, Schmeiser 1980):

Assumption 1 There exists a perfect uniform $(0,1)$, $U(0,1)$, random number generator that can produce a sequence of independent random variables uniformly distributed on $(0,1)$.

Assumption 2 Computers can store and manipulate real numbers.

Although Assumptions 1 and 2 are used for developing RVG theory, the assumptions are violated when implementing RVG algorithms on digital computers. In addition, there are often several methods that have been developed for generating random variates from any particular distribution. The evaluation of a random variate generator from the point of view of design or selection considers the following characteristics (Cheng 1998, Devroye 1986a, Schmeiser 1980):

1. Exactness;
2. Speed;
3. Space; and
4. Simplicity.

Exactness or accuracy refers to how well the generator produces random variates with the characteristics of the desired distribution. This refers to the theoretical exactness of the random variate generator itself, as well as the error that is induced by the $U(0,1)$ random number generator and the error induced by digital computer calculations. *Speed* refers to the computational set-up and execution time required to generate random variates. *Space* refers to computer memory that is required for the generator. Although space is not typically a major consideration for modern computers, computer memory was an important consideration in the early days of RVG development. *Simplicity* refers to the both the simplicity of the algorithm as well as the simplicity of implementation. This includes the number of lines of code, support routines required, number of mathematical operations, as well as portability across platforms and interaction with other simulation methods such as variance reduction techniques. The importance of each of these criteria will vary depending on the particular situation or simulation application.

Devroye (1986b) demonstrates the issues of generating samples of independent and identically distributed random variates with regard to sample independence, consistency, sample indistinguishability, moment matching, and efficiency. The methods presented in this paper have been used extensively to design and evaluate random variate generators. In particular, the importance of this work has made it one of the most frequently cited WSC papers in the history of WSC.

4 RANDOM VARIATE GENERATION CONCEPTS

In this section, the primary concepts on which many random variate generators are developed will be discussed. These basic concepts appear in early work by von Neumann (1951), Yagil (1963), Butler (1956), and Teichroew (1953, 1965).

4.1 The Inverse Transform Method (Inversion)

The inverse transform method (sometimes simply referred to as inversion) utilizes the inverse of the cumulative distribution function (cdf) of the random variable under consideration to generate observations. The general approach is as follows.

Given a random variable X with cdf $F(x)$, generate a $U(0,1)$ random number u corresponding to the u^{th} fractile of the cdf, $F(x) = u$. The value of the random variate generated is the value of x such that,

$$x = F^{-1}(u).$$

For discrete distributions, suppose that X can take the values $\{x_1, x_2, \ldots, x_n\}$ with probabilities $\{p_1, p_2, \ldots, p_n\}$, respectively, such that $\sum_{i=1}^{n} p_i = 1$. The cdf is of the form

$$F(x) = \sum_{i:x_i \leq x} p_i$$

and the inverse is

$$F^{-1}(u) = \min\{x | u \leq F(x)\}.$$

For continuous distributions, the same concept holds. For some continuous distributions, the cdf is directly invertible and a few examples are shown in Table 1. For additional examples of directly invertible cdfs and associated random variate formulations see Cheng (1998), Law (2015), and Banks et al. (2010).

Table 1: Examples of distributions with directly invertible distribution functions (Devroye 2006).

Distribution	Density	Distribution Function	Random Variate
Exponential	e^{-x}, $x>0$	$1-e^{-x}$	$\ln(1/U)$
Weibull	$ax^{a-1}e^{-x^a}$, $a>0$, $x>0$	$1-e^{-x^a}$	$(\ln(1/U))^{1/a}$
Cauchy	$\left(\pi(1+x^2)\right)^{-1}$	$(1/2)+(1/\pi)\arctan x$	$\tan(\pi U)$
Pareto	ax^{-a-1}	$1-x^{-a}$	$U^{-1/a}$

4.2 The Composition Method

Situations arise in which a density function, f can be written as a weighted sum of r other densities (Cheng 1998),

$$f(x)=\sum_{i=1}^{r} p_i f_i(x).$$

where $p_i>0$ and $\sum_{i=1}^{n} p_i=1$. The density f is referred to as a compound or mixture density. Examples include (a) the arrival of customers from multiple sources in a queueing system where the likelihood from each source is known along with the distribution from each source; and (b) the amount of rain on a particular day where there is a probability of whether it rain or not, and if so, the distribution of rainfall. In addition, the use of the composition method arises in situations such as generating random variates from a LaPlace distribution by sampling a random variate from an exponential distribution and then assigning a random positive or negative sign (Schmeiser 1980).

4.3 The Acceptance–Rejection Method

The acceptance–rejection method is often used when a closed-form cumulative distribution function does not exist or is difficult to calculate. In this method, variates are generated from one distribution and are either accepted or rejected in such a way that the accepted values have the desired distribution. Schmeiser (1980) presents the following general acceptance–rejection algorithm.

Given a random variable X, let $f(x)$ denote the desired density function of X. Let $t(x)$ be any majorizing function of $f(x)$ such that $t(x)\geq f(x)$ for all values of x. Let $g(x)=t(x)/c$ denote the density function proportional to $t(x)$ such that $c=\int_{-\infty}^{\infty} t(x)dx$.

1. Generate $x\sim g(x)$.
2. Generate $u\sim U(0,1)$.
3. If $u>f(x)/t(x)$, then reject x and go to step 1.
4. Return x.

The execution time of the the acceptance–rejection algorithm depends on three main factors: (1) the time to generate x from $g(x)$; (2) the time to perform the comparison in step 3; and (3) the number of iterations required to return an accepted value for x.

5 EVOLUTION OF RANDOM VARIATE GENERATORS (1950–PRESENT)

In this section, the evolution of random variate generators are described. The methods are organized by the types of distributions as many of the references focus on particular distributions. This section covers selected continuous and discrete distribution families having primary emphasis in the literature. In each section, an attempt has been made to identify the earliest distribution family specific random variate generator followed by the successive advances and developments. For an excellent survey of random variate generation methods (including a comprehensive list of references) developed prior to 1980, see Schmeiser (1980, 1981).

5.1 Normal Distribution

The early development of generators for normal random variates appears in the 1950s (Box and Muller 1958, Muller 1958). In particular, Box and Muller (1958) present a method for generating a pair of independent random variates from the same normal distribution from a pair of random numbers. Muller (1958) presents an inversion method for generating normal variates that utilizes a stepwise approach, breaking the distribution into sub-intervals and approximating the distribution in each sub-interval using Chebyshev-type polynomials. Muller (1959) compares the methods to other early approaches based on computation time.

Marsaglia and Bray (1964) improve on the polar method of Box and Muller (1958) and are able to gain computational efficiency while producing accurate normally distributed variates.

To further improve the computational efficiency, researchers developed algorithms based on composition and acceptance–rejection methods. In particular, Marsaglia, MacLaren, and Bray (1964) represent the normal density function as the mixture of three densities and utilize a combination of composition and acceptance–rejection to produce a normal random variate generator that is very fast, requires the storage of 300–400 constants, and is accurate.

Ahrens and Dieter (1972) develop alternative methods for generating random variates considering the computational effort of algorithms for the normal and exponential distributions and perform a detailed comparison. In addition, Marsaglia (1964) develops methods to effectively generate variates from the tails of normal distributions.

Finally, Thomas et al. (2007) presents a review of normal (Gaussian) random number generators and compares their accuracy and efficiency.

5.2 Gamma Distribution

According to Schmeiser (1980), the first exact random variate generation method for the gamma distribution was developed by Jöhnk (1964). Gamma random variate generation involving acceptance–rejection methods are presented by Wallace (1974), Ahrens and Dieters (1974), Fishman (1976a), and Tadikamalla (1978) where alternative majorizing functions under various ranges of the shape parameter α are developed in attempt to minimize computational speed.

Given the slow speed of exact algorithms, approximation methods were developed by several authors including Ramberg and Schmeiser (1974), Ramberg and Tadikamalla (1974), and Wheeler (1975). However, the increased speed of these methods is not enough to overcome the loss of accuracy when compared to exact methods.

Exact algorithms with faster execution times have continued to be developed. Marsaglia (1977) and Schmeiser and Lal (1980) present algorithms based on a modification of the acceptance–rejection method referred to as the squeeze method. In these methods a function that is easy to evaluate and is less than the density function $f(x)$ for all x of the gamma variable is used to make quick accept decisions. Cheng and Feast (1979, 1980) focus on creating algorithms for generating gamma variates that are simple to implement and enable an increased range of the shape parameter α.

In addition, Sarkar (1996) develops a procedure that combines the composition method, squeeze method, and aliasing to generate gamma variates with shape parameter greater than 1. Marsaglia and Tsang (2000) present a procedure for generating a gamma variate with shape parameter greater than 1 as the cube of a scaled normal variate.

5.3 Beta Distribution

The beta distribution with shape parameters $\alpha > 0$ and $\beta > 0$ has a density function that can take on many different shapes including a bell shape, J-shape, U-shape, triangular shape, and a skewed bell shapes. In addition, the beta distribution is defined on the unit interval but can be shifted or scaled to cover a specified range. The flexibility of the beta distribution adds complexity to development of variate generation methods. As such, initial methods typically focused on special cases. Fox (1963) generated beta variates for distributions having integer values of α and β.

An exact generation method is presented by Jöhnk (1964) but has increased computation time as α and/or β increase. An acceptance–rejection approach for generating beta variates is given by Dieter and Ahrens (1974). In addition, Cheng (1978) develops a acceptance–rejection type generator for beta variates that uses four exponential function evaluations and two random numbers per trial. For this method, the expected number of trials needed per variate generated is bounded above and the speed of variate generation largely insensitive to the particular values of α and β. Schmeiser and Babu (1980) develop a squeeze-type, acceptance–rejection algorithm that utilizes piece-wise linear and exponential majorizing function and a piece-wise linear minorizing function to reduce the computational time of generating beta variates. Finally, Press et al. (2007) provide efficient code for generating beta variates for all values of α and β.

5.4 Poisson Distribution

To generate discrete Poisson random variates, Kahn (1956) discusses the methods of inversion and acceptance–rejection. Schaffer (1970) implements and algorithm based on the acceptance–rejection approach proposed by Kahn (1956).

Ahrens and Deiter (1974) develop a composition method for generating Poisson variates, and Fishman (1976b) presents an alternative method. However, it is found that both of these increase in computational time with the square-root of the mean.

An exact algorithm that has a stable execution time is first presented by Atkinson (1979). This generator utilizes a modified table look-up procedure. Devroye (1981) develops a method that utilizes the acceptance–rejection method with a normal majorizing function for the main portion of the distribution, the inverse transform method for the tail left tail of the distribution, and an exponential majorizing function for the right tail of the distribution.

Kemp and Kemp (1991) design an algorithm for generating Poisson variates when the mean may change form the generation of one variate to the next. The algorithm utilizes a unidirectional search starting at the mode; the modal probability and modal cumulative probability are calculated using asymptotic approximations.

5.5 Binomial Distribution

To generate binomial random variates, one recognizes immediately that a straightforward algorithm is to sum the results of N Bernoulli trials.

Recognizing the relationship between binomial distribution and the beta distribution, Relles (1972) and Ahrens and Dieter (1974) construct binomial generators that have computational times that increases slowly as N increases.

Kachitvichyanukul and Schmeiser (1988) develop a binomial variate generator using an acceptance–rejection, squeeze method and divides the area under the majorizing function four regions. The center portion of the distribution uses triangular functions for the majorizing and minorizing functions. The tails utilize exponential majorizing functions. The algorithm is accurate and is computationally efficient.

5.6 Other Univariate and Multivariate Distribution Generators

Stadlober and Zechner (1999) construct a patchwork (composition) algorithm that creates rectangles that cover the density of the desired distribution. The algorithm samples to select a rectangle (patch) and then performs acceptance–rejection, if needed, to generate the random variate. This method is applied to both continuous and discrete distributions. The algorithm requires longer set-up times than some other methods, but the marginal variate generation time is relatively fast.

Marsaglia and Tsang (1998) present a novel univariate random variate generation procedure called the Monte Python method. This method gets it name from the opening video of Monte Python's Flying Circus where the method peals off layers of a function that approximates the distribution function until the value form the distribution function is reached (similar to the Monty Python stylized head in the video with a hinged top that folded open, with various items flying out. See, YouTube Video (Monty Python 1974)).

Devroye (1996) presents and develops random variate generation algorithms that consist of only one line of code. The focus is to develop a series of methods that are simple to implement with only one line of code without primary regard to computational speed.

Wagner and Wilson (1996) present a novel, flexible method for modeling and generating univariate Bèzier distributions. The method enables the convenient generation of unimodal and multimodal densities.

Devroye (1997) presents a series of algorithms for generating random variates from multivariate unimodal densities.

Additional generators for univariate and multivariate distributions are presented in the surveys by Schmeiser (1980) and Devroye (2006).

6 GENERATION OF STOCHASTIC POINT PROCESSES (1975-PRESENT)

In addition to univariate and multivariate random variate generation, the generation of stochastic point processes is an important aspect of many simulation studies. These stochastic processes are often used to model events such as the arrival process of customers that may vary with time. Some examples include transactions initiated in a data base system (Lewis and Shedler 1976); the arrival of storms at off-shore oil drilling sites (Lee, Wilson, and Crawford 1991); the arrival of organ transplant patients and donors to organ transplant centers in a simulation study conducted for the United Network for Organ Sharing (Pritsker et al. 1995, Pritsker et al. 1996, Harper et al. 2000); and the arrival of patients in community clinics (Alexopoulos et al. 2008) among many others.

Stochastic arrival processes have often been modeled using nonhomogenerous Poisson processess (NHPP) where the arrival rate $\lambda(t), t > 0$ changes over time and the mean-value function, $\Lambda(t)$, represents the expected cumulative number of arrivals up to time t. Cinlar (1975) utilizes the method of inversion to generate NHPPs. Realizations of an NHPP with a exponential linear exponential rate function (Lewis and Shedler 1978, Lewis and Shedler 1979a) and a degree-two exponential polynomial rate function (Lewis and Shedler 1979b) are generated using thinning methods. Kuhl, Wilson, and Johnson (1997) generate NHPPs having a long-term trend and multiple periodicities by generating exponential random variates with a mean of 1 and then use a search procedure to invert the mean-value function. In addition, Klein and Roberts (1984) and Leemis (1991) generate arrivals from an NHPP with nonparametric rate and mean-value models, respectively.

In addition to NHPPs, nonstationary point processes with dispersion ratios not equal to 1 have also been modeled. Alexopoulos et al. (2008) utilize a nonparametric method that utilizes thinning to generate these processes. Gerhardt and Nelson (2009) produce a renewal process with dispersion ratio greater than or less than 1 utilizing an inversion method and a thinning method. Finally, Liu et al. (2015) present an efficient combined inversion–thinning method for generating nonstationary non-Poisson processes.

7 CONCLUDING REMARKS

Random variate generation has had an extensive and interesting past. This paper only provides some of the highlights of the vast history of random number generation and its importance to the field of simulation. However, even though random variate generation has been studied extensively, opportunities and challenges still exist for developing random variate generations to accurately and efficiently simulate systems to solve problems of ever increasing complexity.

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